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Abstract

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Triangles in Riemannian Spaces with a Pole

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We shall call a point O of a metrically complete, three times continuously differentiable Riemannian space R^m a **pole** if all geodesic rays issuing from O do not intersect at any other point. It is known that in simply connected Riemannian spaces of negative curvature every point is a pole, whence a number of important properties of such spaces follow.

In this note we study Riemannian spaces R^m possessing the following properties:

I. In R^m there exists a pole O.

II. R^m has positive curvature.

From I it follows that any segment of a geodesic with initial point at O is shortest and, consequently, contains no points conjugate to O.

Instead of II it is sufficient to assume that the curvature of R^m is positive only in the two-dimensional directions tangent to the rays issuing from O.

For spaces R^m satisfying the above conditions, the following theorems are proved.

Theorem 1. In every triangle formed by shortest curves, one of whose vertices coincides with the pole, the sum of the angles at the other two vertices is less than π.

Theorem 2. In every triangle in R^m, one of whose vertices coincides with the pole, the excess of the sum of the angles is less than the angle at the pole.

Theorem 3. Let l_1, l_2 be geodesic rays issuing from the pole; let $\rho(s)$ be the distance between the points of these rays lying at distance s from the pole O. Then $\rho(s)$ is a strictly increasing function of s $(0 \leq s < \infty)$.

We note that for two-dimensional surfaces with a pole the above theorems follow easily from the Gauss-Bonnet theorem; however, in the multidimensional case their proof requires the use of more complicated methods.

1. Introduce along some geodesic ray l, issuing from O, a normalized Fermi coordinate system (1). Let the coordinate x^1 coincide with the length measured from O along l, and in what follows denote it, for brevity, by x. Let S be a twice continuously differentiable two-dimensional surface in R^m containing l. Denote by $M_l(x)$ the point of l at distance x from O, and by $u^i(x)$ a unit vector tangent to S at the point $M_l(x)$ and normal to

l; then $u^i(x)$ is continuously differentiable with respect to x. Calculations analogous to those of Synge (^{2}) show that the curvature of S along l is equal to

$$R_{1i,1j}u^iu^j - \sum_{i=1}^{m} u^{i\prime 2},$$

where $R_{ki,lj}$ is the Riemann curvature tensor.

Conversely, if a twice continuously differentiable unit vector $u^i(x)$, normal to l, is given, then the arcs of geodesics tangent ⋯

$u^i(x)$ at the points $M_l(x)$ constitute a twice continuously differentiable surface S, which we shall call the **Sindzha surface**. Denote by $c(x)$ the least curvature of R^m over all two-dimensional directions tangent to l at the point $M_l(x)$.

Lemma 1. Suppose that, for $x_0 \leqslant x < \infty$, there is defined a twice continuously differentiable function $b(x) < c(x)$; let α^i, β^i be vectors satisfying the conditions

$$\sum_{i=1}^{m} \alpha^{i2} = 1, \qquad \sum_{i=1}^{m} \alpha^i\beta^i = 0, \qquad \alpha_1 = \beta_1 = 0, \qquad R_{1i,\,1j}\alpha^i\alpha^j - \sum_{i=1}^{m} \beta^{i2} = b(x_0).$$

Then there exists a unit vector $u^i(x)$ normal to l, satisfying the conditions $u^i(x_0) = \alpha^i$, $u^{i\prime}(x_0) = \beta^i$, and a twice continuously differentiable surface S containing all the points $M_l(x)$ $(x_0 \leqslant x < \infty)$, touching at each point $M_l(x)$ the vector $u^i(x)$ and having at the point $M_l(x)$ curvature $b(x)$.

Proof. Without loss of generality, one may assume that $\alpha^3 > 0$, $\beta^i = 0$ $(i = 3, 4, \dots, m)$; then $\alpha^2 = 0$. Put $u^i(x) = \alpha^i$ $(i = 4, \dots, m)$, $u^2 = \lambda \sin\theta$, $u^3 = \lambda\cos\theta$, where $\lambda = \left(1 - \sum_{i=4}^{m} \alpha_i^2\right)^{1/2}$, and let θ be a solution of the differential equation

$$\frac{d\theta}{dx} = \frac{1}{\lambda}\sqrt{R_{1i,\,1j}u^{iu^{j-b(x)}}},$$

satisfying the initial condition $\theta(x_0) = 0$. It is not hard to prove that $u^i(x)$ is defined, is twice continuously differentiable, and has length 1 for $x_0 \leqslant x < \infty$. We take for S the Sindzha surface corresponding to $u^i(x)$ in the manner indicated above.

Using the expression given above for the curvature of S, it is easy to verify that $u^i(x)$ and S satisfy all the requirements of the lemma.

2. Let K be a twice continuously differentiable cone in R^m, formed by geodesic rays l issuing from the pole O. Introduce on K coordinates x, θ, where x is the distance of the point $M \in K$ from O, and θ is the angle on

the cone between the initial direction of the geodesic OM and some fixed direction. Then the metric of the cone everywhere except the pole has the form $ds^2 = dx^2 + G^2 d\theta^2$, $G(x, \theta) > 0$ (see, for example, (3), p. 166). Let the curvature of K along some ray $\theta = \theta_0$ be $a(x)$ $(0 < x < \infty)$. Then there exists, as can be shown by direct computation, a finite limit of $a(x)$ as $x \to 0$; $\eta(x) = G(x, \theta_0)$ satisfies the Jacobi equation $\eta'' + a(x)\eta = 0$,

$$\lim_{x\to 0} \eta(x) = 0, \qquad \eta(x) > 0 \ (x > 0), \qquad \lim_{x\to 0} \eta'(x) = 1. \tag{1}$$

3. **Lemma 2.** For the function $\eta(x)$ defined in Lemma 1, for $0 < x < \infty$ the inequality $\eta'(x) > 0$ holds.

Proof. Suppose that $x_* > 0$ is the least value of x for which $\eta'(x) = 0$; then from equation (1) it follows that $a(x_*) \geqslant 0$. Let $x_0 \geqslant x_*$ be the point nearest to x_* at which $a(x) = 0$; such a point exists, since otherwise $a(x) > 0$ for $x \geqslant x_*$ and, consequently, $\eta(x)$ vanishes at some point $x' > 0$, contrary to (1). Let $u(x)$ $(0 < x \leqslant x_0)$ be a unit vector tangent to K and orthogonal to l. Put in Lemma 1 $\alpha^i = u^i(x_0)$, $\beta^i = u^{i\prime}(x_0)$; since $a(x_0) = 0$, one may take as $b(x)$ any twice continuously differentiable function on the interval $[x_0, \infty)$ satisfying the conditions $b(x_0) = 0$, $0 < b(x) < c(x)$ $(x_0 < x < \infty)$.

By Lemma 1, $u^i(x)$ is defined on the interval $(0, \infty)$; construct the surface S in the same way as in Lemma 1, tangent at the point $M_1(x)$ to the vector $u^i(x)$ $(0 < x < \infty)$. The solution of equation (1) which coincides with $\eta(x)$ for $x \leqslant x_0$ satisfies the conditions $\eta'(x_*) = 0$, $\eta''(x) < 0$ $(x_* < x < \infty,\ x \neq x_0)$, and therefore vanishes at some point x''. But then, by the necessary Jacobi condition, the segment of the ray l of length $x'' + 1$ is not shortest on S and, all the more, in R^m. Meanwhile, it follows from condition (1) that the rays issuing from O remain shortest along their entire length.

Lemma 3. *The integral curvature of any sector of the cone K with angle at the pole O equal to α is strictly less than α.*

Proof. By the propositions of Sec. 2, for a sector D bounded by a curve $x = x(\theta)$ the following equalities hold:

$$\iint_D a\,ds = \iint_D a(x,\theta)G(x,\theta)\,dx\,d\theta = -\iint_D \eta''_{xx}(x,\theta)\,dx\,d\theta =$$

$$= -\int_0^\alpha d\theta \int_0^{x(\theta)} \eta''_{xx}\,dx = -\int_0^\alpha \eta'_x(x,\theta)\Big|_0^{x(\theta)} d\theta = \alpha - \int_0^\alpha \eta'_x(x(\theta),\theta)\,d\theta,$$

and the lemma follows from Lemma 2.

4. Proof of Theorem 1. Let OM_1M_2 be a triangle made up of shortest curves. Joining O by shortest curves to all points of M_1M_2, we obtain a twice continuously differentiable surface D belonging to the cone K. By Lemma 3, the integral curvature of D is less than the angle α on the cone formed by OM_1 and OM_2. Remove from D the geodesic disk D_ε of radius ε with center at O. Then the Gauss–Bonnet theorem may be applied to the domain $D \setminus D_\varepsilon$. Using the known expression for the geodesic curvature of a geodesic circle (see (^{3}), p. 170), we obtain, as $\varepsilon \to 0$, the relation

$$\text{angle } OM_1M_2 + \text{angle } OM_2M_1 - \pi = \iint_D a\,ds - \alpha,$$

which, by Lemma 3, is negative. Thus Theorem 1 is proved. The proof of Theorem 2 follows directly from Theorem 1.

5. Proof of Theorem 3. Let l_1, l_2 be geodesic rays issuing from the pole; let $M_1(s), M_2(s)$ be points of the rays l_1, l_2 at distance s from O. Then $\rho(s) = \rho(M_1(s), M_2(s))$. Suppose that $\rho(s)$ is not a strictly increasing function. Then there is an $s_0 > 0$ for which $\rho(s)$ attains a relative maximum. Take a monotonically decreasing sequence of values $s_n \to s_0$. Then, by choosing a subsequence, one may obtain shortest curves $M_1(s_n)M_2(s_n)$ converging to some shortest curve $M_1(s_0)M_2(s_0)$. By Theorem 1, the sum of the angles α_1, α_2 formed by this shortest curve $M_1(s_0)M_2(s_0)$ with $OM_1(s_0)$, $OM_2(s_0)$, is less than π. Therefore $\cos\alpha_1 + \cos\alpha_2 > 0$. But simple considerations from the calculus of variations show that

$$\rho(s_n) - \rho(s_0) = (s_n - s_0)(\cos\alpha_1 + \cos\alpha_2) + O(s_n - s_0),$$

so that for sufficiently large n, $\rho(s_n) > \rho(s_0)$, which contradicts the definition of the number s_0. Theorem 3 is proved.

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Note: Figure translations are in progress. See original paper for figures.

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