# SELF-ADJOINT EXTENSIONS OF A SYSTEM OF COMMUTING SYMMETRIC OPERATORS

1960

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## Abstract

## Full Text

## MATHEMATICS

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# SELF-ADJOINT EXTENSIONS OF A SYSTEM OF COMMUTING SYMMETRIC OPERATORS

*(Presented by Academician S. L. Sobolev on 29 III 1960)*

Consider a linear topological space $\Phi$, in which commuting linear continuous operators $A$ and $B$ act; let $H$ be the Hilbert space obtained from $\Phi$ by completion with respect to the continuous nondegenerate scalar product $(x, y)$, and suppose that the operator $\bar{A}$ (the closure in $H$ of the operator $A$) is self-adjoint in $H$, while the operator $B$ is symmetric. Does there exist a self-adjoint extension of the operator $B$ that commutes with $\bar{A}$? (Self-adjoint operators are called commuting if their spectral families commute.) Below a solution of this problem is given, with applications to representations of positive-definite functionals.

**Definition 1.** We shall say that the operator $\bar{A}$ is **strongly self-adjoint** in $H = \bar{\Phi}$ if $\Phi$ contains a subset $\Phi_1$ such that: a) $\Phi_1$ is dense in $\bar{\Phi}$ in the topology of $\Phi$; b) if $x \in \Phi_1$, then in the subspace $L_x$ spanned by the elements $\{A^k x,\ k = 0, 1, 2, ...\}$, the operator $A$ has a self-adjoint closure.

A strongly self-adjoint operator is, obviously, self-adjoint in $H$ (in the usual sense).

I. Denote the set $(B - \lambda E)\Phi$ by $\Phi^\lambda$, and let $H^\lambda = \overline{\Phi^\lambda}$. Obviously, $A\Phi^\lambda \subseteq \Phi^\lambda$ and the operator $A$ is symmetric in $H^\lambda$.

**Theorem 1.** *Suppose the following conditions are satisfied: 1) the operator $A$ is strongly self-adjoint in $H^\lambda$ for some $\lambda$ with $\operatorname{Im}\lambda \neq 0$; 2) in $\Phi$ there is an involution $x \leftrightarrow x^*$, and $(x, y) = (x^*, y^*)$ for all $x, y \in \Phi$, and the operators $A, B$ are real with respect to this involution, i.e. $Ax^* = (Ax)^*$.*

*Then the operator $B$ has a self-adjoint extension commuting with $\bar{A}$.*

**Proof.** If the operator $\bar{A}$ is strongly self-adjoint in some $H^{\lambda_0}$ ($\operatorname{Im}\lambda_0 \neq 0$), then this is true also for any $\lambda$, $\operatorname{Im}\lambda \neq 0$; indeed, $H^{\lambda_0}$ is mapped onto $H^\lambda$ one-to-one and continuously by the operator

$$V_{\lambda_0}^{\lambda} = \frac{\bar{B} - \lambda E}{\bar{B} - \lambda_0 E},$$

and moreover

$$V^{\lambda}_{\lambda_0}(A^k x) = A^k(V^{\lambda}_{\lambda_0} x) \quad (x \in \Phi^{\lambda_0}).$$

It may therefore be assumed that $\bar{A}$ is strongly self-adjoint in $H^i$. Let $\Phi^i = (B - iE)\Phi$ and $\Phi_1 \subseteq \Phi^i$ be the set introduced in Definition 1. If $x$ is any element of $\Phi$, $h = Bx + ix$, $g = Bx - ix$, then from $\operatorname{Im}(A^{nBx}, x) = 0$ one can obtain that

$$(A^n h, h) = (A^n g, g)$$

or

$$\int \lambda^n \, d(E_\lambda h, h) = \int \lambda^n \, d(E_\lambda g, g) = c_n.$$

If $g \in \Phi^i_1$ (in other words, $x \in (B - iE)^{-1}\Phi^i_1$), then from the last equality we obtain

$$(E_\lambda h, h) = (E_\lambda g, g). \tag{1}$$

Indeed, the condition $g \in \Phi^i_1$ entails the determinacy of the moment problem (see (1)) $\int \lambda^n d\sigma(\lambda) = c_n$; hence (1) follows. Substituting, instead of $h$ and $g$, $Bx + ix$ and $Bx - ix$, we obtain from (1)

$$\operatorname{Im}(E_\lambda Bx, x) = 0. \tag{2}$$

Since $\Phi^i_1$ is dense in $\Phi^i$, (2) is true for all $x \in (B - iE)^{-1}\Phi^i = \Phi$. From (2) it follows that the operator $E_\lambda B$ is symmetric in $\Phi$.

Let $\overline{B}$ be the closure of $B$ in $H$, and let

$$V = \frac{\overline{B} + iE}{\overline{B} - iE}, \quad (\overline{B} + iE)D_{\overline{B}} = \Delta_V,$$

$$(\overline{B} - iE)D_{\overline{B}} = D_V.$$

From what has been said it follows that for all $x, y \in D_V$ the equality

$$(E_\lambda x, y) = (E_\lambda Vx, Vy) \tag{3}$$

holds.

Denote $R^+ = H \ominus D_V$ and $R^- = H \ominus \Delta_V$. All $E_\lambda$ map $R^+$ (respectively $R^-$) into itself. We must find a unitary extension $\widetilde{V}$ of the operator $V$, mapping

$R^+$ onto $R^-$ in such a way that for any $x, y \in R^+$ an equality analogous to (3) holds:

$$(E_\lambda x, y) = (E_\lambda \widetilde{V} x, \widetilde{V} y). \tag{4}$$

For this purpose we choose in $R^+$ such an orthonormal basis $\{x_i\}$ that all functions $\varphi_{ij}(\lambda) = (E_\lambda x_i, x_j)$ are real for $-\infty < \lambda < \infty$. (This can be done as follows: decompose $R^+$ into mutually orthogonal subspaces, in each of which the operator $A = \int \lambda \, dE_\lambda$ has simple spectrum; each such subspace $H_i$ with the operator $A$ is realized as the space $L^2(\sigma_i)$, and the operator $\widetilde{E}^i_\Delta$ becomes multiplication by the characteristic function of the interval $\Delta$; now choose in each $L^2(\sigma_i)$, isomorphic to $H_i$, an orthonormal basis of real functions.) We now construct the desired extension by setting $\widetilde{V} x_i = x_i^*$ ($x_i^*$ is the element involutive to $x_i$). Since $E_\lambda$ is a real operator, we have $(E_\lambda x_i^*, x_j^*) = ((E_\lambda x_i)^*, x_j^*) = \overline{(E_\lambda x_i, x_j)}$; but $(E_\lambda x_i, x_j)$ is real for all $\lambda$; therefore $(E_\lambda x_i^*, x_j^*) = (E_\lambda x_i, x_j)$, i.e. (4) is fulfilled, as required.

Usually one has to consider not one scalar product, but an entire family of scalar products in $\Phi$, in each of which the operators $A$ and $B$ are symmetric. It is therefore necessary that the conditions of Theorem 1 be preserved for all these scalar products. Let us consider one case where this occurs.

**Definition 2.** An element $x \in \Phi$ will be called **Carleman** if there exists a numerical sequence $m_k$ such that

$$\sum m_{2k}^{-1/2k} = \infty,$$

and the sequence

$$\left\{ \frac{1}{m_k} A^k x \right\}$$

is bounded in the topology of $\Phi$.

**Definition 3.** The operator $A$ in $\Phi$ will be called **Carleman** if $\Phi$ contains a dense set of Carleman elements.

Denote by $\Phi^K$ the set of Carleman elements in $\Phi$. Since $B - \lambda E$ is a continuous operator commuting with $A$, we have $(B - \lambda E)\Phi^K \subseteq \Phi^K$. Therefore the operator $A$ in $\Phi^\lambda$ is also Carleman. From the well-known Carleman criterion for the determinacy of the moment problem (see (1)) we conclude that $\overline{A}$ is strongly self-adjoint in each $H^\lambda$.

Therefore the following is true:

**Theorem 1*.** *If $A$ is a Carleman operator in $\Phi$ and the operators $A$ and $B$ are real with respect to the involution in $\Phi$ (where $(x^*, y^*) = \overline{(x, y)}$), then $B$ admits a self-adjoint extension in $H$ commuting with $\overline{A}$.*

II. Applications.

Let $\Phi$ be a nuclear algebra or a closed linear subspace of a nuclear algebra with a (unique) characteristic operator $A$ (defined–

conditions, see ($^2$), more precisely ($^3$)); $T(x)$ is a positive-definite functional on $\Phi$. As was shown by A. G. Kostyuchenko and B. S. Mityagin, $T(x)$ admits a (unique) expansion

$$T(x) = \int \chi_\lambda(x)\, d\sigma(\lambda)$$

in the eigenfunctionals $\chi_\lambda$ of the operator $A$, if the latter has in $H_T$ (this denotes the Hilbert space obtained from $\Phi$ by completion in the norm

$$\|x\| = \sqrt{T(x \circ x^*)}$$

) a self-adjoint closure. We shall introduce one class of such spaces.

**Definition 4.** We shall call a nuclear algebra (or a subspace of a nuclear algebra) $\Phi$ a **Carleman space** if $\Phi$ possesses a unique characteristic operator $A$, and the latter is Carleman in $\Phi$. In addition, throughout it is assumed that $\Phi$ is a space with involution.

Every positive-definite functional on the Carleman space $\Phi$ admits a unique expansion in the eigenfunctionals of the characteristic operator $A$; in other words, the closure $\overline{A}$ of the operator $A$ in the space $H_T$ is always a self-adjoint operator. This follows from the fact that $A$ is strongly self-adjoint in $H_T$.

**Theorem 2.** Let $\Phi = \Phi_1 \otimes \Phi_2$ be the tensor product of nuclear algebras (or linear subspaces of algebras) ($^2$,$^3$ ) $\Phi_1$ and $\Phi_2$ with characteristic operators $A_1$ and $A_2$, and suppose that $\Phi_1$ is a Carleman space.

Then every positive-definite functional $T(x)$ on $\Phi$ admits an expansion (in general, nonunique) in common eigenfunctionals of the operators $A_1$ and $A_2$.

**Proof.** If the operator $A_1$ in $\Phi_1$ is Carleman, then the operator $A_1 \otimes E_2$ ($E_i$ is the identity operator in $\Phi_i$) will be Carleman in $\Phi$. By Theorem 1, the operator $E_1 \otimes A_2$ has in $H_T$ a self-adjoint extension commuting with $\overline{A_1 \otimes E_2}$. By the theorem of Kostyuchenko and Mityagin, $T(x)$ admits the required representation.

In what follows, a space $\Phi$ with a (unique) characteristic operator $A$ will be denoted by $(\Phi, A)$.

**Theorem 3.** Let $\Phi$ be a space of functions absolutely integrable on $(-\infty, \infty)$, with continuous convolution

$$\varphi \circ \psi = \int \varphi(x-t)\psi(t)\, dt$$

and with characteristic operator

$$c\,\frac{d^k}{dx^k},$$

and suppose that a continuous "dilation" operator

$$K\varphi(x) = \varphi(nx)$$

is defined in $\Phi$ for all large $n$.

Then, if $\Phi$ contains at least one Carleman element $\varphi_0(x)$ and

$$\int \varphi_0(x)\,dx \neq 0,$$

then $\Phi$ is a Carleman space.

**Proof.** The sequence

$$\varphi_n(x) = n\varphi_0(nx)$$

is an "identity" in $\Phi$, i.e.

$$\varphi_n(x) \circ \varphi(x) \to \varphi(x)$$

for all $\varphi \in \Phi$ (we assume that

$$\int \varphi_0(x)\,dx = 1$$

). But all $\varphi_n(x)$ are, obviously, Carleman elements, as are the elements $\varphi_n(x) \circ \varphi(x)$. Thus, for any $\varphi(x) \in \Phi$ we have found a sequence of Carleman elements converging to $\varphi(x)$, as was required.

Verification that $\Phi$ is a Carleman space is reduced by this theorem to a certain quasi-analytic problem.

**Theorem 4.** The spaces

$$\left(S,\, i\frac{d}{dx}\right), \quad \left(S_\alpha^\beta,\, i\frac{d}{dx}\right) \quad \text{for } \alpha+\beta \geq 1,$$

$$\left(+S_\alpha^\beta,\, \frac{d^2}{dx^2}\right) \quad \text{for } \alpha+\beta \geq 1,\ \alpha \geq \frac{1}{2},$$

are Carleman spaces*.

**Proof.** We shall prove, for example, that

$$\left(+S_\alpha^\beta,\, \frac{d^2}{dx^2}\right)$$

for

$$\alpha+\beta \geq 1, \quad \alpha \geq \frac{1}{2}$$

is a Carleman space. Denote by $G$ and $G_1$

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* For the notation see ($^1$), Ch. 4, and ($^3$), § 4.

respectively, to the regions $(\alpha+\beta \geq 1,\ \alpha \geq 1/2)$ and $(\alpha+\beta \geq 1,\ \beta \leq 1/2)$. If the point $(\alpha,\beta) \in G_1$, then directly from the inequalities defining ${}^+S_\alpha^\beta$ one can show that all functions $\varphi(x) \in {}^+S_\alpha^\beta$ are Carleman. If $(\alpha,\beta) \notin G_1$, i.e. $\beta > 1/2$, then one can find a point $(\alpha',\beta') \in G_1$ such that $\alpha' \leq \alpha$ and $\beta' \leq \beta$; then ${}^+S_{\alpha'}^{\beta'} \subset {}^+S_\alpha^\beta$. But all elements of ${}^+S_{\alpha'}^{\beta'}$ are Carleman. It follows from Theorem 3 that ${}^+S_\alpha^\beta$ is also a Carleman space.

It follows from Theorem 4 that positive-definite functionals on the indicated spaces admit a unique representation.

**Theorem 5 (on the extension of a positive-definite function from a strip).** *A positive-definite function $f(x_1,\dots,x_n)$, continuous in the strip $-\infty < x_i < \infty$ $(i=1,2,\dots,n-1)$, $-h \leq x_n \leq h$, of n-dimensional space, admits an extension to the whole plane by the formula*

$$f(x_1,\dots,x_n) = \int_{R_n} \exp\left[i\sum x_k\xi_k\right]\, d\sigma(\xi_1,\dots,\xi_n), \tag{5}$$

*where $\sigma(\xi_1,\dots,\xi_n)$ is a bounded, nonnegative measure.*

**Proof.** $f(x_1,\dots,x_n)$ may be regarded as a functional on the space of functions $S(\pi_h)$ that vanish outside the strip

$$\pi_h = \{x:\ -\infty < x_i < \infty\ (i=1,2,\dots,n-1),\ -h \leq x_n \leq h\}$$

and satisfy the estimates

$$\left|D^k\varphi(x_1,\dots,x_n)\right| \leq C_{k,p}|x|^{-p}, \quad \text{where } k=(k_1,\dots,k_{n-1}),$$

$$D^k = \partial^{k_1+\dots+k_{n-1}}/\partial x_1^{k_1}\cdots\partial x_{n-1}^{k_{n-1}}, \quad |x| = (x_1^2+\dots+x_n^2)^{1/2}.$$

Obviously,

$$S(\pi_h) = \underbrace{\hat{S}\hat{\otimes}\cdots\hat{\otimes}\hat{S}}_{n-1 \text{ times}}\hat{\otimes}K_h,$$

where $K_h$ is the space of finitely infinitely differentiable functions on $(-h,h)$. The space $S$ with characteristic operator $i\frac{d}{dx}$ is a Carleman space (Theorem 4). Theorem 1 is applicable (obviously, it remains valid if, instead of one operator $A$, one takes any number of Carleman operators $A_i$ $(i=1,2,\dots,n)$, commuting with one another and commuting with $B$): the operator $i\frac{\partial}{\partial x_n}$ admits a self-adjoint extension commuting with the closures of the operators

$$i\frac{\partial}{\partial x_k} \quad (k=1,2,\dots,n-1).$$

By the Kostyuchenko-Mityagin theorem ($^2$), $f(x_1,\dots,x_n)$ admits the expansion (5) in common eigenfunctions

$$\chi_{\xi_1\cdots\xi_n} = \exp\left[i\sum x_k\xi_k\right]$$

of the operators

$$i\,\frac{\partial}{\partial x_k} \quad (k = 1, 2, \dots, n).$$

Theorem 5 is proved.

I express my gratitude to B. S. Mityagin for his attention to this work.

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Received
25 III 1960

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*Note: Figure translations are in progress. See original paper for figures.*

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