MATHEMATICS

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Abstract

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ON SOME PROBLEMS IN THE ANALYTIC THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

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§ 1. In the present work we study analytic solutions of one class of partial differential equations:

$$\frac{\partial^r u}{\partial x^r} = ay^m \frac{\partial^{m+1}}{\partial y^{m+1}}, \tag{1}$$

where r and m are natural numbers, and a is a constant number. The class (1) includes the heat equation ($r = 2,\ m = 0$), and also the equation of mixed type

$$\frac{\partial^2 u}{\partial x^2} = ay \frac{\partial^2 u}{\partial y^2}. \tag{2}$$

Throughout the whole work we shall consider solutions of equation (1) that are entire in (x, y), i.e., functions $u(x, y)$ analytic in any bicylinder $|x| \leq R,\ |y| \leq R_1$. From equation (1) it is not difficult to obtain the general form of such an expansion:

$$u(x,y) = \sum_{l=0}^{m-1} P_l(x) y^l + f_m(x) y^l + \sum_{k=1}^{\infty} y^{m+k} \frac{f_m^{(kr)}(x)(k-1)!(k-2)!\dots 2!}{a^k (m+k)!(m+k-1)!\dots(m+1)!}. \tag{3}$$

Here the functions $P_l(x)$, $l = 0, 1, \dots, m-1$, must be polynomials of degree not exceeding $r - 1$, while the function $f_m(x)$ is an entire function whose growth is restricted by the requirement of uniform convergence of the series (3) in any bicylinder $|x| \leq R,\ |y| \leq R_1$.

For this condition to be satisfied, it is necessary and sufficient that the function $f_m(x)$ be an entire function of order

$$\frac{1}{1 - (m+1)/r}$$

and of zero type, if $m+1<r$, and an arbitrary entire function, if $m+1\geq r$. We shall denote the class of such functions by $H_{m,r}$. Every class $H_{m,r}$ contains all entire functions of exponential type. The function $\partial^m u/\partial y^m\big|_{y=0}$ will be called the growth function of the given solution.

Theorem 1. *In order that the function* $u(x,y)$*, defined by the series (3), be an entire in* (x,y) *solution of equation (1), it is necessary and sufficient that its growth function belong to the class* $H_{m,r}$*.*

The Cauchy problem with respect to the variable y for equation (1) in the class of solutions entire in x,y can be formulated as follows:

Find a solution of equation (1) satisfying the conditions:

1) $\partial^s u/\partial y^s\big|_{y=0}=P_s(x)$, $s=0,1,\dots,m-1$; $P_s(x)$ are polynomials of degree $\leq r-1$;
2) $\partial^m u/\partial y^m\big|_{y=0}=f(x)$, where $f(x)\in H_{m,r}$.

The Cauchy problem has a unique solution in the class of solutions entire in (x,y) for any prescribed polynomials $P_s(x)$ $(s=0,1,\dots,m-1)$, of degree $\leq r-1$, and for any entire function $f(x)$ from $H_{m,r}$.

§ 2. For analytic solutions of equation (1) one can also pose certain new problems. We shall consider here one such problem (we shall call it problem A).

Problem A. Let $y=\mu(x)\equiv\sum_{i=0}^{l}A_i x^i$ be the equation of a parabola of order $l<r$. It is required to find an entire solution in (x,y) of equation (1) such that:

a) $\partial^s u/\partial y^s\big|_{y=0}=P_s(x)$, $\quad s=0,1,\dots,m-1$, where $P_s(x)$ are given polynomials of degree not exceeding $r-1$;
b) $u(x,\mu(x))=\lambda(x)$, where $\lambda(x)$ is a given entire function.

The function $\lambda(x)$ cannot be prescribed completely arbitrarily. From representation (3), putting $y=\mu(x)$, we obtain that the function

$$\nu(x)=\frac{\lambda(x)-\sum_{l=0}^{m-1}[\mu(x)]^l P_l(x)}{[\mu(x)]^m} \tag{4}$$

must be entire. The solution of problem A is reduced to the determination of the function $f_m(x)$ from the condition

$$\nu(x)=f_m(x)+\sum_{k=1}^{\infty}[\mu(x)]^k\frac{2!\,3!\dots(k-1)!}{(m+1)!(m+2)!\dots(m+k)!}\,\frac{f_m^{(kr)}(x)}{a^k} \tag{5}$$

or

$$\nu(x) = Z(x) + \sum_{k=1}^{\infty} Q_k(x) Z^{(k)}(x),$$

where $Z(x) = f_m(x)$, and $Q_k(x)$ is a polynomial of degree $\leq k-1$, i.e., to the solution of a differential equation of infinite order of a definite type considered in (^{1}).

Theorem 2. Let $u(x, y)$ be an entire solution in (x, y) of equation (1). Then:

1) the functions $\partial^s u/\partial y^s|_{y=0} = P_s(x), \quad s = 0, 1, \dots, m-1$, must be polynomials of degree not exceeding $r-1$;
2) if $\lambda(x)$ is the value of the solution $u(x, y)$ on the parabola $y = \mu(x) \equiv \sum_{k=0}^{l} A_k x^k,\ l < r$, then the function $\nu(x)$, defined by equality (4), must be entire.

Conversely, for any parabola $y = \mu(x)$ of order $< r$ one can indicate a certain subclass G of sufficiently slowly growing entire functions ($G \subset H_{m,r}$) such that, whatever the prescribed polynomials $P_0, P_1, \dots, P_{m-1}$ of degree $\leq r-1$ and whatever the prescribed entire function $\lambda(x)$, provided only that $\nu(x) \in G$, the solution of problem A always exists. This solution is unique in the class of entire solutions in (x, y) of equation (1) of sufficiently slow growth (namely, in the class of functions $u(x, y)$ such that their growth functions $\partial^m u/\partial y^m|_{y=0}$ belong to G).

Using the results of paper (^{1}), one can give a method for the approximate solution of problem A (with an estimate of the error), and also show the correct dependence of the solution (in a certain metric) on the data of the problem.

The class G is defined by specifying the parabola $y = \mu(x)$ and, generally speaking, changes in passing from one parabola of order $< r$ to another. However, it always contains the class y_0 of entire functions $f(x)$ of zero order such that

$$\sum_{m=0}^{\infty} |f^{(m)}(0)| \left(\prod_{k=0}^{m} (k!) \right) < \infty$$

(in this case, as a rule, the class y is substantially wider than the class G_0).

We shall give some concrete results for special types of parabolas.

1. $\mu(x) \equiv Cx^l,\ 0 \leq l < r$. Introduce the auxiliary entire function

$$F_m(x) = \sum_{s=1}^{\infty} \frac{x^s m!(m-1)! \dots 2!}{s!(s+1)! \dots (s+m)!}$$

and denote by d_0 the unique positive root of the equation $F_m(x) = 1$. Further, by G_C denote the class of entire functions of order $1 - \frac{l}{r}$ and type

$$< \frac{r}{r-l}\left(\frac{d_0}{C}\right)^{1/r}.$$

Theorem 3. Let $u(x,y)$ be a solution of problem A for the parabola $y = Cx^l$, $l < r$. Then:

1) the functions $\partial^s u/\partial y^s|_{y=0}$, $s = 0,1,\dots,m-1$, will be polynomials of degree not exceeding $r-1$;

2) the function $\lambda(x)$ has the form

$$\lambda(x) = \sum_{k=0}^{m-1} C^k x^{lk} P_k(x) + \varphi(x)x^{lm},$$

where $\varphi(x)$ is an entire function.

Conversely, if the function $\lambda(x)$ satisfies condition 2) and belongs to G_C, then the solution of problem A exists and is unique in the class of entire-in-(x,y) solutions whose functions of growth belong to G_C.

2. $\mu(x) \equiv C$. Let $G_{m,r}$ be the class of entire functions of the first order of finite type, for which none of the Borel-associated singularities of the functions coincides with zeros of the function

$$1 + \sum_{k=1}^{\infty} \frac{C^k 2!3!\dots(k-1)!\,x^{kr}}{(m+1)!(m+2)!\dots(m+k)!}.$$

Using (2), we arrive at the following result:

Theorem 4. The problem of determining a solution $u(x,y)$ of equation (1) under the conditions:

1) $\partial^s u/\partial y^s|_{y=0} = P_s$, $s = 0,1,\dots,m-1$; P_s are polynomials of degree $\le r-1$;

2) $u(x,c) = \lambda(x)$, $\lambda(x) \in G_{m,r}$,

is solvable and, moreover, uniquely so in the class of entire-in-(x,y) solutions $v(x,y)$ whose functions of growth belong to $G_{m,r}$.

Theorems 2-4 ensure the existence and uniqueness of the solution of problem A in the class of entire-in-(x,y) solutions of equations (1) of sufficiently slow growth. It is necessary to note that in the whole class of entire-in-(x,y) solutions (defined by the condition that the functions of growth belong to $G_{m,r}$) uniqueness, generally speaking, will no longer hold.

If the solution of equation (1) is determined by prescribing it on a parabola of order $\ge r$, then the study of problem A becomes considerably more complicated, and the nature of the results is different. For the solvability of problem A it is necessary that the function $\lambda(x) = u(x, \mu(x))$ satisfy, in addition to the natural conditions of type (4) (for analytic solutions), further conditions of an algebraic character. Uniqueness of the solution of problem A no longer holds in any class of entire functions $v(x, y)$ of arbitrarily slow growth that contains all polynomials.

§ 3. Consider some particular cases of equation (1).

1. For equation (2), problem A consists in determining the solution from its values on the straight lines $y = 0$ and $y = Cx + B$: $u(x, 0) = \lambda_1(x)$, $u(x, Cx + B) = \lambda(x)$. For the solvability of this problem, by Theorem 2, it is necessary that the function $\lambda_1(x)$ be linear: $\lambda_1(x) = ax + b$, and that the function $\lambda(x)$ be entire, with $\lambda(-B/C) = -aB/C + b$. The last condition is the natural compatibility condition (coincidence of the values of the functions $\lambda_1(x)$ and $\lambda(x)$ at $x = -B/C$). Let d_0 be the positive root of the equation $J_1(2i\sqrt{x}) = 2i\sqrt{x}$. From Theorem 3 it follows:

Theorem 5. The problem of determining an entire-in-(x, y) solution of equation (2) from the data $u(x, 0) = ax + b$; $u(x, Cx + B) = \lambda(x)$ is always solvable if the function $\lambda(x)$ is an entire function of order 1/2 and type

$< 2\sqrt{d_0/|Ca|}$, with $\lambda(-B/C) = -aB/C + b$. The solution is unique in the class of functions $v(x, y)$ for which $\partial v/\partial y|_{y=0}$ is an entire function of order 1/2 and type $< 2\sqrt{d_0/|Ca|}$.

If $C = 0$, we arrive at the following problem: to find a solution of equation (2) such that $u(x, 0) = ax + b$, $u(x, d) = \lambda(x)$. Here the line $y = d$ may lie both in the hyperbolic and in the elliptic part of the plane (depending on the sign of d). By Theorem 4 the problem is solvable if $\lambda(x)$ is an entire function of exponential type for which no singularity of the Borel-associated function coincides with the zeros of the entire function $1 + \sum_{k=1}^{\infty} \frac{d^k}{a^k} \frac{x^{2k}}{k!(k+1)!}$. The solution is unique in the class of entire functions $v(x, y)$ in (x, y) such that $\partial v/\partial y|_{y=0} \in G_{1,2}$.

2. Let us also consider the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial y$. Problem A consists in determining a solution from its values on the line $y = Cx + B$. By the linear change $x = x_1 - B/C$ we reduce this problem to the following: to find a solution of the heat equation such that $u(x, Cx) = \lambda(x)$, where $\lambda(x)$ is a given entire function. Denoting by G_C the class of entire functions of order 1/2 and type $< 2\sqrt{\ln 2/|C|}$, and applying Theorem 3, we obtain:

Theorem 6. *Problem A, under the condition that $\lambda(x) \in G_C$, is uniquely solvable in the class of solutions entire in (x, y) whose growth functions belong to G_C.*

In connection with Theorem 6 it is appropriate to note one result of Holmgren (^{3}), who showed that a regular (i.e., continuous together with first-order derivatives) integral $v(x, y)$ of the heat equation in the whole plane is uniquely determined by prescribing on the line $y = Cx$, $C \neq 0$, the values $v(x, Cx)$ and $\partial v(x, Cx)/dx$. Therefore there exists an infinite set of regular integrals $v(x, y)$ taking a prescribed value $\lambda(x)$ on the line $y = Cx$. At the same time, as follows from Theorem 6, if $\lambda(x)$ is a sufficiently smooth entire function ($\lambda(x) \in G_C$), then in the class of entire integrals in (x, y) of sufficiently slow growth the specification of the function $\lambda(x) = u(x, Cx)$ uniquely determines the corresponding solution.

Problem A for the heat equation admits a clear physical interpretation. As is known, the heat equation characterizes the propagation of temperature in an infinite thin thermally insulated rod. Suppose that along the rod, with constant velocity $\alpha = dx/dt$, there moves an instrument which at each instant of time t determines the temperature of the rod at the corresponding point $x = \alpha t$ (it is assumed, for definiteness, that at $t = 0$ the instrument is at the point $x = 0$). It is required, from the readings of the instrument—the function $\varphi(x)$—to determine the law of propagation of temperature in the rod. It follows from Theorem 6 that this problem is solvable if $\varphi(x)$ is an entire function of order 1/2 and type $< 2\sqrt{|\lambda| \ln 2}$. The solution is unique in the class of sufficiently slowly growing entire functions in (x, t) (namely, such that $v(x, 0)$ is an entire function of order 1/2 and type $< 2\sqrt{|\lambda| \ln 2}$).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.