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Abstract

Full Text

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DENSELY EMBEDDED IDEALS OF SEMIGROUPS

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An ideal A of a semigroup S is called **densely embedded** in S ([1,2]) if it satisfies the following conditions:

a) Every nontrivial* homomorphism of the semigroup S induces** a nontrivial homomorphism of the semigroup A.

b) If T is any semigroup containing S as a proper subsemigroup in such a way that A is an ideal of T, then there exists a nontrivial homomorphism of the semigroup T inducing an isomorphism on A.

E. S. Lyapin ([1,2]) and the author ([3]) used the concept of a densely embedded ideal to obtain an internal characterization of semigroups of transformations.

In the present note these results are transferred to abstract semigroups satisfying very weak restrictions (Theorem 4). Under these restrictions it is proved that every semigroup A can be a densely embedded ideal of some (unique up to isomorphism) semigroup S; the connection of this semigroup S with semigroups of shifts is indicated (Theorem 2). A number of other properties of densely embedded ideals are obtained; in particular, it is shown that every isomorphism of the semigroup A can be uniquely extended to an isomorphism of the semigroup S containing A as a densely embedded ideal (Theorems 6 and 7).

1. A **left shift** of a semigroup A is a transformation ψ (a mapping into itself) of the set A such that

$$\forall_{x,y\in A}\ \psi(xy) = \psi x \cdot y \ \ *** .$$

A **right shift** of a semigroup A is a transformation φ of the set A such that

$$\forall_{x,y\in A}\ (xy)\varphi = x \cdot y\varphi$$

(unlike left shifts, we shall write right shifts as right operators on the set A). The set $\Phi(A)$ of all right shifts of the semigroup A and the set $\Psi(A)$ of all left shifts of the semigroup A are subsemigroups of the semigroup $S(A)$ of all transformations of the set A.

Shifts $\varphi \in \Phi(A)$, $\psi \in \Psi(A)$ are called **linked** ([5]) if

$$\forall_{x,y \in A}\ (x\varphi)y = x(\psi y).$$

Let T be an arbitrary semigroup containing A as an ideal, and let t be any element of T. The mappings φ_t, ψ_t of the semigroup A into itself

$$\forall_{x,y \in A}\ \{x\varphi_t = xt, \quad \psi_t y = ty\}$$

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* Not an isomorphism.

** Let A be a subset of a set S; f a single-valued mapping of the set S onto some set S'; φ such a mapping of the set A that $fa = \varphi a$ for every $a \in E$. It is said that the mapping f **induces** the mapping φ.

*** See the notation in the article ([4]).

are, obviously, associated translations of the semigroup A. For $T = A$, the mappings φ_t and ψ_t are called the internal (respectively right and left) translations of the semigroup A generated by the element $t \in A$ [(6)].

2. Let $\Phi(A) \times \Psi(A)$ be the direct product of the semigroups $\Phi(A)$ and $\Psi(A)$, i.e., the set of all pairs (φ, ψ), where $\varphi \in \Phi(A)$, $\psi \in \Psi(A)$, with operation:

$$(\varphi, \psi)(\varphi', \psi') = (\varphi\varphi', \psi\psi').$$

Denote by A_i the subset of the semigroup $\Phi(A) \times \Psi(A)$ consisting of all pairs (φ_a, ψ_a), where $a \in A$; by S_A the subset of $\Phi(A) \times \Psi(A)$ consisting of all pairs (φ', ψ'), where φ' and ψ' are associated translations of the semigroup A (see item 1). Obviously, A_i is a subsemigroup of the semigroup $\Phi(A) \times \Psi(A)$.

Lemma 1. S_A is a maximal subsemigroup of the semigroup $\Phi(A) \times \Psi(A)$ containing A_i as an ideal.

3. Elements a, a' of the semigroup A are called **equiacting** if

$$\forall x \in A\{xa = xa', \quad ax = a'x\}.$$

Everywhere below in the present note, semigroups without equiacting elements are considered.

Lemma 2. Let A be a semigroup without equiacting elements; T an arbitrary semigroup containing A as an ideal. The mapping f of the semigroup T into the semigroup S_A (see items 1, 2),

$$\forall t \in T \quad ft = (\varphi_t, \psi_t)$$

is a homomorphism inducing an isomorphism on the semigroup A.

4. From Lemmas 1 and 2 there follows the following theorem:

Theorem 1. If A is a semigroup without equiacting elements, then the semigroup A_i is a densely embedded ideal of the semigroup S_A (see item 2).

Let A_l be the subsemigroup of all internal left translations of the semigroup A (see item 1); $\Psi^*(A)$ the subsemigroup of $\Psi(A)$ consisting of all left translations $\psi \in \Psi(A)$ for which there exist associated right translations $\varphi \in \Phi(A)$. From Theorem 1 there follows the following assertion, which generalizes the result of item 3.3.1 of article (3):

If A is a semigroup without left equiacting elements (3), then A_l is a densely embedded ideal of the semigroup $\Psi^*(A)$.

5. An ideal A of a semigroup S is called its d-ideal (3) if it satisfies condition a) and

b) For every semigroup T containing A as an ideal, there exists a homomorphism into the semigroup S inducing the identity isomorphism on the semigroup A.

It turns out that the semigroup A_i is a d-ideal of the semigroup S_A. From the isomorphism of the semigroups A and A_i, and also from the properties of a d-ideal (see Theorem 1.5 of article (3)), the theorems 2–4 below follow.

Theorem 2. Let A be a semigroup without equiacting elements. There exists a semigroup S containing A as a densely embedded ideal. Every semigroup S' containing A as a densely embedded ideal is isomorphic to the semigroup S_A.

In article (3) it is shown that every d-ideal of a semigroup S is densely embedded in S. The converse of this theorem is the following result:

Theorem 3. If a densely embedded ideal A of a semigroup S contains no equiacting elements, then A is a d-ideal of the semigroup S.

Theorem 3 together with Theorem 1.5 of article (3) leads to the following theorem:

Theorem 4. Let A be a semigroup without equiacting elements, S a semigroup containing A as a densely embedded ideal.

In order that the semigroup S' be isomorphic to the semigroup S, it is necessary and sufficient that S' contain a densely embedded ideal A', isomorphic to A.

6. Property a) of densely embedded ideals can be strengthened as follows:

Theorem 5. Let a semigroup A without right identities be a densely embedded ideal of a semigroup S; let T be an arbitrary subsemigroup of the semigroup S containing A.

Every nontrivial homomorphism of the semigroup T induces a nontrivial homomorphism of the semigroup A.

The following theorem may be regarded as the converse of Theorem 5:

Theorem 6. Let A be a semigroup without right identities; let f be an arbitrary isomorphism of the semigroup A onto some semigroup A'. Suppose, further, that A is a densely embedded ideal of a semigroup S; A' is an ideal of a semigroup T' such that every nontrivial homomorphism of the semigroup T' induces a nontrivial homomorphism of the semigroup A'.

There exists, and moreover is unique, an isomorphism f' of the semigroup T' into S which is an extension of the isomorphism f.

In particular, from Theorem 6 there follows the following assertion, strengthening Theorem 4:

Let f be an isomorphism of a semigroup A without right identities onto some semigroup A'; let S and S' be semigroups containing respectively A and A' as densely embedded ideals. There exists, and moreover is unique, an isomorphism f' of the semigroup S onto S' which is an extension of the isomorphism f.

Putting $A = A'$, $S = S'$, we obtain from this:

Theorem 7. Let A be a semigroup without right identities, which is a densely embedded ideal of a semigroup S. Every automorphism of the semigroup A can be extended, and moreover in a unique way, to an automorphism of the semigroup S.

7. Let the ideal A of the semigroup T satisfy condition a) of Sec. 1. From Theorems 2 and 6 there follows the possibility of isomorphically embedding the semigroup T into some semigroup containing A as a densely embedded ideal. For Σ-semigroups [(4)] this gives the following result:

Theorem 8. Let A be a Σ-semigroup containing no right identities. If A is a Σ-ideal of the semigroup S, then A is also a Σd-ideal of S.

8. From Theorem 4, from the corollary to Theorem 1 given in Sec. 4, and also from the results of the papers [(1,2)], there follows the following theorem (it can also be proved directly):

Theorem 9. Let Ω be an arbitrary nonempty set; let A be a semigroup with multiplication

$$\forall_{x,y \in A}\ xy = x,$$

having the same cardinality as the set Ω; let $S(\Omega)$ be the semigroup of all transformations of the set Ω.

The semigroup $S(\Omega)$ is isomorphic to the semigroup $\Psi(A)$ of all left shifts of the semigroup A.

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Note: Figure translations are in progress. See original paper for figures.