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Abstract

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ON CONDITIONS FOR COMPACTNESS OF A FAMILY OF SURFACES OF FINITE TOPOLOGICAL TYPE

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Let:

a) the surfaces of the family $\{T\}$ lie in a bounded part of the ambient space;

b) the areas $A(T)$ of the surfaces of the family are bounded in the aggregate.

These conditions, analogous to the conditions of the theorem given by Hilbert for curves, do not ensure compactness of the family $\{T\}$ with respect to the Fréchet metric. As is shown in this note, if one restricts oneself to families of nondegenerate surfaces satisfying conditions a), b), then for compactness of the family it is necessary and sufficient that the natural condition of uniform nondegeneracy be fulfilled. The results find application in the calculus of variations.

All surfaces under consideration $T : x = f(u),\ u \in D \subset R^2,\ x \in \mathfrak{R}$, have finite topological type (H, r). This means that the domain of variation of the parameters D is homeomorphic to a surface of genus H with r boundary curves. The case $r = 0$, corresponding to families of closed surfaces, is not excluded. All criteria are formulated for families of orientable, but not oriented, surfaces. The orientation of surfaces, as well as nonorientability, introduces nothing essentially new. As the canonical domain of variation of the parameters one considers the unit square $K = [0, 1;\ 0, 1]$ with a prescribed identification of the boundary segments. All vector functions $x = f(u)$ are absolutely continuous in the sense of Tonelli, and their partial derivatives are square integrable. The ambient space $\mathfrak{R}$ is a manifold with a Riemannian metric. Identification of parametric representations is defined as in ([1]) by means of elementary operations of one-dimensional subdivision and coarsening and piecewise-smooth transformations of the domain D. The Fréchet distance $r(T_1, T_2)$ between the surfaces $T_1 : x = f_1(u),\ u \in D_1;\ T_2 : x = f_2(u),\ u \in D_2$, is equal to

$$\inf_{f_1, f_2} \rho(f_1, f_2), \qquad \rho(f_1, f_2) = \sup_{u \in D} |f_1(u), f_2(u)|,$$

where the lower bound is taken over all possible parametric representations of the surfaces in one domain; $|x_1, x_2|$ is the distance between two points of the manifold $\mathfrak{R}$, defined as the lower bound of the curves joining the points x_1, x_2. The above-adopted definition of identification may now be replaced by the condition: $T_1 \equiv T_2$, if $r(T_1, T_2) = 0$. However, what follows does not depend on this extension of the notion of identification.

We shall call a curve $\Gamma_0 : \gamma_0$ a section of the surface $T : f(u),\ u \in D$, if Γ_0 is defined by a piecewise-smooth closed Jordan curve γ_0 lying inside D, or by a Jordan arc γ_0 lying inside D, excluding the ends.

A surface **degenerates** if there exists a sequence of sections $\{\Gamma_n : \gamma_n\}$ for which

$$d(\Gamma_n) = \operatorname{osc}\{f_n, \gamma_n\} \to 0 \quad (n \to \infty),$$

moreover, either the sections Γ_n do not split the surface T (degeneration of the type of the surface), or each section Γ_n splits T into parts

$$T_n^i :\ x = f(u),\ u \in D,\ i = 1, 2; \qquad D_n^1 + D_n^2 = D - \gamma_n,$$

for which $d(T_n^i) > \alpha > 0,\ i = 1, 2;\ n = 1, 2, \dots,\ \alpha$ does not depend on n (decomposition of the surface T).

1. Theorem 1. Let $\{T\}$ be a closed family of nondegenerate surfaces of type (H, r), satisfying conditions a), b).

For compactness of the family $\{T\}$ it is necessary and sufficient that the following condition be fulfilled:

c) for every $\varepsilon > 0$ there exists a $\delta > 0$, independent of the surface of the family, such that for any section Γ_0, from $d(\Gamma_0) < \delta$ it follows that Γ_0 splits the surface T, and one of the parts has diameter less than ε.

The proof of the necessity of condition c) for compactness of the family $\{T\}$ is elementary.

Let us explain the proof of sufficiency by the example of a family of surfaces $\{T\}$ of spherical type. Let $\{T_n\} \subset \{T\}$. We may assume that $d(T_n) > \beta > 0$. The sequence $\{T_n : f_n\}$ can be replaced by a sequence of polyhedra $\{P_n : h_n\}$, for which there exist piecewise-conformal parametric representations ((1), pp. 86-90). Choosing on $\{P_n\}$ asymptotically nonbranching cuts tt^{-1} ((1), pp. 71-72) with lengths not less than $\beta/2$, one can normalize the parametric representations in the canonical domain so that the three-point condition is fulfilled ((1), p. 73). The equicontinuity of these parametric representations is proved as in ((1), pp. 74-76), and then the proof of sufficiency is completed as in ((1), pp. 88-90).

2. Theorem 2. Let $\{T\}$ be a closed family of nondecomposing surfaces, generally speaking of different topological types (H, r), $H \le H_0,\ r \le r_0$, satisfying conditions a), b).

For compactness of the family $\{T\}$ it is necessary and sufficient that the following condition be fulfilled:

c_1) for every $\varepsilon > 0$ there exists a $\delta > 0$, independent of the surface of the family, such that for any section Γ_0, from $d(\Gamma_0) < \delta$ it follows that, if Γ_0 splits the surface T, then one of the parts has diameter less than ε.

Conditions c), c_1) express uniform nondegeneration (respectively, nondecomposition) of the surfaces of the family. The inequality $d(\Gamma_0) < \delta$ in these conditions may be replaced by the inequality $l(\Gamma_0) < \delta$, where $l(\Gamma)$ is the length of the curve Γ.

The proof of Theorem 2 reduces to the choice of a subsequence $\{T_{n_k}\}$, satisfying the conditions of Theorem 1, from the subsequence $\{T_n\} \subset \{T\}$. The choice of $\{T_{n_k}\}$ is connected with the splitting of the surfaces of the sequence $\{T_n\}$, which is described in detail in ((1), pp. 91-94).

3. Let $\{T\}$ be a family of surfaces of type $(H, 0)$ or a family of surfaces of type (H, r), for which the lengths of the boundary curves are uniformly bounded.

Theorem 3. For compactness of the family $\{T\}$, satisfying conditions a), b), it is sufficient that the following condition be fulfilled:

c_2) for every $\varepsilon > 0$ there exists a $\delta > 0$, independent of T, such that any section Γ_0, $l(\Gamma_0) < \delta$, splits the surface T, and one of the parts T_1 has the type of a disk and satisfies the "isoperimetric inequality" $A(T_1) \leq q\{l(\Gamma_0)^2\}$, where q does not depend on T.

Remark. If the assumption that the lengths of the boundary curves of the family are uniformly bounded is dropped, then for the validity of Theorem 3 it is sufficient to supplement it with the following condition:

c_3) for every $\varepsilon > 0$ there exists a $\delta > 0$, independent of T, such that

for each part $T_1 \subset T$ of disk type with boundary Γ_1, consisting of arcs, $\Gamma' \subset \Gamma$ and Γ'' lying inside T, from $d(\Gamma_1') < \delta$, $l(\Gamma_1') < \delta$ it follows that $A(T_1) < \varepsilon$ (here Γ is the boundary curve of the surface).

The sufficiency of conditions c_2), c_3) for the fulfillment of condition c_1) is shown with the aid of the known estimates ([1]), pp. 77-78. This proves Theorem 3.

4. We shall say that a surface $T : x = f(u)$, $u \in D$, of type (H, r) splits N times if there exists a sequence of finite systems of nonintersecting sections

$$\Gamma_{nk} : \gamma_{nk}, \quad k = 1, 2, \ldots, N_1; \quad n = 1, 2, \ldots; \quad N \leqslant N_1 \leqslant N + H,$$

dividing the surface T into $N+1$ parts $T_{nk} : x = f(u)$, $u \in D_n^k$, $k = 1, 2, \ldots, N+1$, such that $d(\Gamma_{nk}) \to 0$, $k = 1, 2, \ldots, N_1$ $(n \to \infty)$, $d(T_n^k) = \operatorname{osc}\{f, D_n^k\} > \alpha > 0$ for all n, k.

Theorem 4. *Let $\{T\}$ be a closed family of surfaces of types (H, r), $H \leqslant H_0$, $r \leqslant r_0$, splitting no more than N times, and satisfying conditions a), b).*

For compactness of the family $\{T\}$ it is necessary and sufficient that the following condition be fulfilled:

c_4) *for any $\varepsilon > 0$ there exists a $\delta > 0$, independent of T, such that for any system of nonintersecting sections $\Gamma_1, \dots, \Gamma_N$ dividing T into $N+2$ parts, from $l(\Gamma_k) < \delta$, $k = 1, 2, \dots, N_1$, it follows that at least one of the parts thus obtained has diameter less than ε.*

For the proof we split the sequence $\{T_n\} \subset \{T\}$ into no more than $N+1$ uniformly nonsplitting sequences, in the same way as is done in ([1]), pp. 91–94.

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REFERENCES

[1] A. G. Sigalov, *Uspekhi Mat. Nauk*, **12**, No. 1, 53 (1957).

Note: Figure translations are in progress. See original paper for figures.