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Abstract

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ON UNIFORMLY CONTINUOUS MAPPINGS OF UNIFORM COMPLEXES INTO A SPHERE

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The present note is devoted to a generalization of the well-known classification theorem of Hopf for mappings of n-dimensional polyhedra into the n-sphere. Below we set forth facts that make it possible to classify uniformly continuous mappings of uniform triangulations* of dimension n into the n-sphere.

Basic definitions. A chain $x = \sum \alpha_i T_i$ of a complex K will be called **uniform** if, for every i, the inequality $\alpha_i < \alpha$** holds, where α is a constant independent of i. A ∇-cycle will be called **uniformly homologous to zero** if it is the ∇-boundary of some uniform chain. Uniform ∇-cycles, as well as uniform ∇-cycles uniformly homologous to zero (of the same dimension), form groups. The quotient group of the uniform ∇-cycles by the subgroup of cycles uniformly homologous to zero will be called the **uniform ∇-group** of the complex K and denoted by the symbol $\nabla^n_u(K)$. Uniformly continuous mappings f_0 and f_1 of a set A into a set B will be called **uniformly homotopic** if, in the direct product (A, I) of the set A with the unit interval I, there exists a uniformly continuous mapping F into B, for every $a \in A$, satisfying the condition $F(a, 0) = f_0(a)$, $F(a, 1) = f_1(a)$***. The set of all uniformly continuous mappings of a set A into a set B decomposes into uniformly homotopic classes.

Main theorem. *Between the set of elements of the group $\nabla^n_u(\widetilde{K}^n)$ of a uniform complex K^n and the set of uniformly homotopic classes of the set of uniformly continuous mappings of the triangulation $\widetilde{K}^n$ into the sphere S^n there exists a one-to-one correspondence.*

Everything that follows is devoted to the proof of this fact.

Lemma 1. *For every uniformly continuous mapping f of a uniform triangulation $\widetilde{K}_1$ into a uniform triangulation $\widetilde{K}_2$, there exist simplicial mappings (simplicial approximations) into $\widetilde{K}_2$ of barycentric subdivisions of corresponding order of the triangulation $\widetilde{K}_1$, uniformly homotopic to the mapping f.*

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* A triangulation (or complex) is called uniform if: 1) the lengths of all its one-dimensional simplexes are bounded above; 2) the distances between any two simplexes of this triangulation that do not have common vertices are bounded

below by some constant $h > 0$ (see the fundamental work on the theory of uniform complexes ([1])).

** In this work all chains and, consequently, all cycles are assumed to be integral; thus all α_i are integers.

*** By the symbol (A, θ) we denote the "layer" of the set (A, I) corresponding to the numerical coordinate θ $(0 \leq \theta \leq 1)$; analogously, $(a, \theta) \in (A, I)$ is the point with coordinates $a \in A$ and θ $(0 \leq \theta \leq 1)$.

This lemma follows from the following facts:

I. The stars of vertices (principal stars) of a uniform triangulation $\widetilde{K}$ form a Lebesgue covering of the set $\widetilde{K}^*$ (see ([2])).

II. Every simplicial mapping of a uniform triangulation $\widetilde{K}_1$ into a uniform triangulation $\widetilde{K}_2$ is uniformly continuous.

III. The barycentric subdivision of a uniform triangulation is itself a uniform triangulation.

IV. Suppose that on the "bases" $(A, 0)$ and $(A, 1)$ of the set (A, I) there are defined mappings, respectively, f_0 and f_1 into E^N, and that

$$\rho[f_0(a, 0); f_1(a, 1)] < l$$

(l is a constant and a is any point of A). If f_0 and f_1 are uniformly continuous, then the mapping F of the set (A, I), defined as follows, is also uniformly continuous:

$$\overrightarrow{OF}(a, \theta) = \overrightarrow{Of_0}(a, 0) + \theta\,[\overrightarrow{Of_1}(a, 1) - \overrightarrow{Of_0}(a, 0)]$$

(O is some fixed point of the space E^N).

Lemma 2. For every uniformly continuous mapping f of a uniform triangulation $\widetilde{K}^n$ into the sphere S^n, there exists a mapping of the triangulation $\widetilde{K}^n$ into S^n, uniformly homotopic to the mapping f, which sends the $(n-1)$-dimensional skeleton of the triangulation $\widetilde{K}^n$ to a single point of the sphere S^n.

We shall call such mappings **special**.

Let φ be a special uniformly continuous mapping of $\widetilde{K}^n$ onto the oriented boundary S^n of some $(n+1)$-dimensional simplex, and let $\widetilde{K}'^n$ be a barycentric subdivision of the triangulation $\widetilde{K}^n$ of such order that the image of each principal star of the triangulation $\widetilde{K}'^n$ under the mapping φ is contained in some principal star of the triangulation S^n. **Define on $\widetilde{K}'^n$ a simplicial approximation g uniformly homotopic to the mapping φ. Under the mapping g, the image of each oriented n-simplex $T_i^n \in \widetilde{K}'^n$ will be a cycle $\gamma_i S^n$ of the sphere S^n. The ∇-cycle**

$$Z_\varphi = \sum_i \gamma_i T_i^n$$

of the complex K^n is called the degree of the mapping** φ^***. Let**

$$x^n = \sum_i \alpha_i T_i^n$$

be a chain of the complex K^n. The number

$$\gamma_\varphi(x^n) = \sum_i \alpha_i \gamma_i$$

is called the degree of the mapping of the chain x^{n**}.

Theorem 1. The degree of a uniformly continuous mapping of a uniform triangulation $\widetilde{K}^n$ into the sphere S^n is a uniform ∇-cycle.

Theorem 2. Every uniform ∇-cycle Z^n of a uniform complex K^n determines a special uniformly continuous mapping of the triangulation $\widetilde{K}^n$ into the sphere S^n, and the cycle Z^n itself is the degree of this mapping.

Lemma 3. Let uniformly homotopic mappings f_0 and f_1 into the sphere S be defined on a subset B of a set A. If one of these mappings, for example f_0, has a uniformly continuous extension F_0 to the set A, then the mapping f_1 also extends to some mapping F_1 of the set A into the same sphere S, uniformly homotopic to the mapping F_0.

Theorem 3. If special mappings φ_0 and φ_1 of a uniform triangulation $\widetilde{K}^n$ into the sphere S^n are uniformly homotopic, then the degrees of these mappings are uniformly homologous.

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* A covering λ is called a **Lebesgue** covering if there exists a covering $\{c_i\}$ such that the covering $\{O(\eta, c_i)\}$ is inscribed in λ, $\eta > 0$.

** The reality of such an assumption is guaranteed by the uniformity of the triangulation $\widetilde{K}^n$ and the uniform continuity of the mapping φ.

*** γ_i does not depend on the choice of subdivisions (see, for example, ([3])).

**** Only those chains are meant for which the degree defined in this way exists.

Indeed, represent the set $(\widetilde{K}^n, I)$ as a prism $\Pi\widetilde{K}^n$ over the triangulation $\widetilde{K}^n$, and on its bases $(\widetilde{K}^n, 0)$ and $(\widetilde{K}^n, 1)$ define respectively the mappings φ_0 and φ_1. Lemma 3 permits us to assume that the mapping Φ, which realizes a uniform homotopy of the mappings φ_0 and φ_1, sends the entire $(n-1)$-dimensional skeleton of the prism $\Pi\widetilde{K}^n$ to one point of the sphere S^n. We define the $(n-1)$-dimensional chain x^{n-1}, which realizes a uniform homology of the degrees of the mappings φ_0 and φ_1, on each simplex $T^{n-1} \in \widetilde{K}^n$ to be the degree of the mapping $\gamma_\Phi(\Pi, T^{n-1})$ of the prism ΠT^{n-1}. Note that the uniform discontinuity of the mapping Φ implies the uniformity of the chain constructed.

Theorem 4. *If the degrees of uniformly continuous special mappings φ_0 and φ_1 of the uniform triangulation $\widetilde{K}^n$ into S^n are uniformly homologous, then φ_0 and φ_1 are uniformly homotopic.*

Proof. Consider the mapping of the sum of the upper and lower bases, coinciding with φ_0 on the lower base and with φ_1 on the upper. Next we construct a uniformly continuous extension to the prism over the $(n-1)$-dimensional skeleton of the triangulation $\widetilde{K}^n$. We extend the resulting mapping to the entire n-dimensional skeleton of the prism $\Pi\widetilde{K}^n$ (denote this extension by Φ). In this case the boundary of each simplex T_i^{n+1} of the prism $\Pi\widetilde{K}^n$ is mapped into S^n inessentially, and the mapping Φ sends the entire $(n-1)$-dimensional skeleton of the prism ΠK^n to one point. Construct for Φ a simplicial approximation G, based on a certain suitable barycentric subdivision of the n-dimensional skeleton of the prism $\Pi\widetilde{K}^n$. Let T^n be some (for example, regular) complex whose boundary is subdivided by a barycentric subdivision of the same order as the n-dimensional skeleton of the prism $\Pi\widetilde{K}^n$, and let T_i^{n+1} be an arbitrary simplex of the prism $\Pi\widetilde{K}^n$. Assigning to the vertices of the simplex T_i^{n+1} bijectively the vertices of the simplex T^{n+1}, to the center of gravity of the simplex T_i^{n+1} the center of gravity of the simplex T^{n+1}, to the centers of gravity of the faces of the simplex T_i^{n+1} the centers of gravity of the corresponding faces of the simplex T^{n+1}, and step by step doing the same for barycentric subdivisions of higher orders, we define linearly a simplicial homeomorphism u_i of the simplex T_i^{n+1} onto the simplex T^{n+1}. Consider the system of mappings $g_i = Gu_i^{-1}$ of the boundary of the simplex T^{n+1}. This system splits into a finite number of classes, each of which contains identically coinciding mappings. The inessentiality of the mappings g_i permits each mapping g_i to be extended to a mapping $\overline{G}_i$ into S^n of the entire simplex T^{n+1}, and the system $\{\overline{G}_i\}$ (just as $\{g_i\}$) contains only a finite number of essentially distinct classes, and therefore (just as $\{g_i\}$) is a uniformly continuous system of mappings. Putting, for every point $a \in T_i^{n+1}$, $F(a) = \overline{G}_i[u_i(a)]$, we obtain the required extension of the mapping G to the entire prism $\Pi\widetilde{K}^n$. Applying Lemma 3, we uniformly continuously extend to ΠK^n also the mapping Φ. The theorem is proved.

Thus, by assigning to each homotopy class the element of the group $\nabla_u^n(K^n)$ containing the degree of some mapping from the given homotopy class, we construct the required one-to-one correspondence.

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Note: Figure translations are in progress. See original paper for figures.