ON THE FLOW OF A CONDUCTING VISCOUS FLUID BETWEEN TWO POROUS PLANES

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Abstract

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ON THE FLOW OF A CONDUCTING VISCOUS FLUID BETWEEN TWO POROUS PLANES

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In the present paper we consider the problem of the flow of a viscous incompressible electrically conducting fluid in the space between two parallel porous planes, when a homogeneous constant external magnetic field H_0 is applied perpendicular to the planes. The flow between porous planes without the influence of a magnetic field has been studied in works ([1–3]).

Let the principal flow of the fluid take place parallel to the planes. The fluid simultaneously enters through one plane and leaves through the other, so that the amount of fluid entering and leaving through the pores remains unchanged along the flow.

We direct the axis Ox along the planes, and the axis Oz perpendicular to them. If the corresponding velocity components are denoted by v_x and v_z, then in the case under consideration one may assume that they depend only on z and t. Then from the continuity equation $\operatorname{div} \mathbf{v} = 0$ it immediately follows that $v_z = v_0(t)$, where $v_0(t)$ is the value of the velocity at the boundary of the plane. The same also applies to the longitudinal field H_x, arising as a result of the motion of the fluid.

The fundamental equations of magnetohydrodynamics ([4,6]) in the case under consideration have exact solutions:

$$v_x = v(z,t), \qquad v_z = v_0(t), \qquad H_x = H(z,t), \qquad H_z = H_0 = \text{const},$$

where v and H satisfy the equations

$$\frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial z^2} = \frac{H_0}{4\pi\rho}\frac{\partial H}{\partial z} - v_0(t)\frac{\partial v}{\partial z} + A(t),$$

$$\frac{\partial H}{\partial t} - \lambda \frac{\partial^2 H}{\partial z^2} = H_0 \frac{\partial v}{\partial z} - v_0(t)\frac{\partial H}{\partial z} \tag{1}$$

where $A(t) = \frac{1}{\rho}\frac{\partial p}{\partial x}$ is prescribed; λ is the coefficient of magnetic viscosity; ν is the kinematic coefficient of viscosity; p is the ordinary pressure; ρ is the density. In this case the functions v and H satisfy the boundary conditions

$$v(z,0) = v^0(z), \qquad H(z,0) = H^0(z),$$

$$v(z,t)\big|_{z=\pm a} = 0, \qquad H(z,t)\big|_{z=\pm a} = 0, \tag{2}$$

where $2a$ is the distance between the porous planes, and the plane $z = 0$ is located midway between them. The condition for H follows from the invariability of the external magnetic field H_0, while the tangential component of $\mathbf{H}$ at the boundary must be continuous.

The solution of this problem reduces to a system of integro-differential equations ([2,5])

$$v(z,t) = \Phi(z,t) + \int_0^t d\tau \int_{-a}^{a} \left(\frac{H_0}{4\pi\rho}\frac{\partial H}{\partial \eta} - v_0(\tau)\frac{\partial v}{\partial \eta}\right) G(z,\eta,t-\tau)\, d\eta,$$

$$H(z,t) = F(z,t) + \int_0^t d\tau \int_{-a}^{a} \left(H_0\frac{\partial v}{\partial \eta} - v_0(\tau)\frac{\partial H}{\partial \eta}\right) G(z,\eta,t-\tau)\, d\eta, \tag{3}$$

where $\Phi(z,t)$ and $F(z,t)$ satisfy the heat-conduction equations:

$$\frac{\partial \Phi}{\partial t} - \nu\frac{\partial^2 \Phi}{\partial z^2} = A(t), \qquad \frac{\partial F}{\partial t} - \lambda\frac{\partial^2 F}{\partial z^2} = 0 \tag{4}$$

and the boundary conditions (2), while $G(z,\eta,t)$ is the Green' s function of the heat-conduction equation

$$G(z,\eta,t) = -\frac{1}{2\sqrt{\pi n t}}\exp\left[-\frac{(z-\eta)^2}{4nt}\right] + g(z,\eta,t),$$

where $g(z,\eta,t)$ is the solution of the heat-conduction equation

$$\frac{\partial g}{\partial t} - n\frac{\partial^2 g}{\partial z^2} = 0, \qquad n = \lambda, \nu,$$

which vanishes at the initial instant and satisfies the boundary conditions

$$g(a,\eta,t) = \frac{1}{2\sqrt{\pi n t}}\exp\left[-\frac{(a-\eta)^2}{4nt}\right],$$

$$g(-a,\eta,t)=\frac{1}{2\sqrt{\pi nt}}\exp\left[-\frac{(-a-\eta)^2}{4nt}\right].$$

Hence it is clear that the determination of F,Φ,g is reduced to a system of Volterra integral equations of the second kind with a regular kernel.

If instead of (3) we consider the system with parameter δ and differentiate it once with respect to z, then for determining $\partial v/\partial z$ and $\partial H/\partial z$ we obtain

$$\frac{\partial v}{\partial z}=\frac{\partial\Phi}{\partial z}+\delta\int_0^t d\tau\int_{-a}^{a}\left[\frac{H_0}{4\pi\rho}\frac{\partial H}{\partial\eta}-v_0(\tau)\frac{\partial v}{\partial\eta}\right]\frac{\partial G}{\partial z}\,d\eta,$$

$$\frac{\partial H}{\partial z}=\frac{\partial F}{\partial z}+\delta\int_0^t d\tau\int_{-a}^{a}\left[H_0\frac{\partial v}{\partial\eta}-v_0(\tau)\frac{\partial H}{\partial\eta}\right]\frac{\partial G}{\partial z}\,d\eta. \qquad (5)$$

We shall seek the functions $\partial v/\partial z$ and $\partial H/\partial z$ in the form of a series

$$\frac{\partial v}{\partial z}=\sum_m^\infty\delta^m W_m,\qquad\frac{\partial H}{\partial z}=\sum_m^\infty\delta^m U_m. \qquad (6)$$

To determine the terms of the series we obtain the recurrence formulas

$$W_0=\frac{\partial\Phi}{\partial z},\qquad U_0=\frac{\partial F}{\partial z},$$

$$W_m=\int_0^t d\tau\int_{-a}^{a}\left[\frac{H_0}{4\pi\rho}U_{m-1}-v_0(\tau)W_{m-1}\right]\frac{\partial G}{\partial z}\,d\eta,$$

$$U_m=\int_0^t d\tau\int_{-a}^{a}\left[H_0W_{m-1}-v_0(\tau)U_{m-1}\right]\frac{\partial G}{\partial z}\,d\eta.$$

It is easy to show that the inequalities

$$|U_m|,\ |W_m|<M(2NK)^m t^{m/2}\frac{\Gamma^m(1/2)}{\Gamma(m/2+1)}, \qquad (7)$$

hold, where M,N,K are constants satisfying the conditions

$$\left|\frac{\partial F}{\partial z}\right|,\ \left|\frac{\partial\Phi}{\partial z}\right|<M,\qquad|H_0|,\ |v(t)|<N,\qquad\int_{-a}^{a}\sqrt{t-\tau}\left|\frac{\partial G}{\partial z}\right|d\eta<K,$$

and Γ is Euler's gamma-function. From (7) it is clear that the series (6) converge absolutely and uniformly when $t < \infty$. For $\delta = 1$, the series (6) gives the solution of our problem.

If $\upsilon_0(t) = 0$, then from (3) we obtain the solution of the nonstationary problem of flow in the space between two plane-parallel solid planes

$$\upsilon(z,t) = \Phi(z,t) + \int_0^t d\tau \int_{-a}^{a} \frac{H_0}{4\pi\rho}\frac{\partial H}{\partial \eta} G(z,\eta,t-\tau)\, d\eta,$$

$$H(z,t) = F(z,t) + \int_0^t d\tau \int_{-a}^{a} H_0 \frac{\partial \upsilon}{\partial \eta} G(z,\eta,t-\tau)\, d\eta.$$

The solution of the stationary problem could have been obtained from the corresponding nonstationary problem considered above by passing to the limit $t \to \infty$, but this limiting transition involves considerable mathematical difficulties; therefore we shall consider the stationary problem directly.

In this case the magnetohydrodynamic parameters υ, H, p do not depend explicitly on time, and (1), (2) take the form:

$$\nu \frac{d^2\upsilon}{dz^2} + \frac{H_0}{4\pi\rho}\frac{dH}{dz} - \upsilon_0 \frac{d\upsilon}{dz} = 0,$$

$$\lambda \frac{d^2 H}{dz^2} - \upsilon_0 \frac{dH}{dz} + H_0 \frac{d\upsilon}{dz} = 0; \tag{8}$$

$$\upsilon(z)\Big|_{z=\pm a} = 0, \qquad H(z)\Big|_{z=\pm a} = 0. \tag{9}$$

The solution of equation (8) under the conditions (9) can be written in the form

$$\upsilon = \frac{A\upsilon_0}{\beta} z + \frac{A}{\nu(k_2 - k_1)} \left\{ B_2 \left(\operatorname{cth} k_1 a - \frac{\operatorname{ch} k_1 z + \operatorname{sh} k_1 z}{\operatorname{sh} k_1 a} \right) - \right.$$
$$\left. - B_1 \left(\operatorname{cth} k_2 a - \frac{\operatorname{ch} k_2 z + \operatorname{sh} k_2 z}{\operatorname{sh} k_2 a} \right) \right\},$$

$$H = -\frac{4\pi A}{H_0} \left\{ \left(\frac{\upsilon_0^2}{\beta} + 1 \right) z + \frac{a}{\nu(k_2 - k_1)} \left[B_2 C_1 \left(\operatorname{cth} k_1 a - \frac{\operatorname{ch} k_1 z + \operatorname{sh} k_1 z}{\operatorname{sh} k_1 a} \right) - \right.\right.$$
$$\left.\left. - B_1 C_2 \left(\operatorname{cth} k_2 a - \frac{\operatorname{ch} k_2 z + \operatorname{sh} k_2 z}{\operatorname{sh} k_2 a} \right) \right] \right\},$$

where

$$C_1 = (v_0 - \nu k_1), \qquad C_2 = (v_0 - \nu k_2), \qquad \beta = \frac{H_0^2}{4\pi\rho} - v_0^2,$$

$$B_1 = -\frac{v_0 a}{\beta} C_1 + \left(\frac{v_0^2}{\beta} + 1\right), \qquad B_2 = -\frac{v_0 a}{\beta} C_2 + \left(\frac{v_0^2}{\beta} + 1\right),$$

$$k_{1,2} = \frac{v_0(\lambda + \nu) \mp \sqrt{v_0^2(\lambda - \nu)^2 + \frac{\lambda\nu}{\pi\rho} H_0^2}}{2\lambda\nu}.$$

If one considers steady flow between two solid planes, then $v_0 = 0$, and we obtain the known solution found by Hartmann ([6,7]):

$$v = \bar{v}_0 \frac{\operatorname{ch} ka - \operatorname{ch} kz}{\operatorname{ch} ka - 1}, \qquad H = -\frac{\bar{v}_0 \, 4\pi\sqrt{\sigma\eta}}{c} \frac{\frac{z}{a} \operatorname{sh} ka - \operatorname{sh} kz}{\operatorname{ch} ka - 1},$$

where

$$k = k_1 = -k_2, \qquad \bar{v}_0 = \frac{A}{\nu k} \frac{\operatorname{sh} ka}{\operatorname{ch} ka - 1}.$$

For $v_0 = 0$ and

$$ak = \frac{aH_0}{c}\sqrt{\frac{\sigma}{\eta}} \ll 1$$

one obtains the value

$$v = \bar{v}_0 \left(1 - \frac{z^2}{a^2}\right),$$

which coincides with the result of ordinary hydrodynamics ([8]).

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Note: Figure translations are in progress. See original paper for figures.