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A. I. VOLPERT

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Abstract

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A. I. VOLPERT

ON THE INDEX OF BOUNDARY-VALUE PROBLEMS FOR A SYSTEM OF HARMONIC FUNCTIONS WITH THREE INDEPENDENT VARIABLES

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The present paper is devoted to the problem of the index of boundary-value problems for elliptic systems of equations with three independent variables, posed by I. M. Gel' fand ([1]). For the problem (I) given below it is shown that its index, generally speaking, is different from zero and can have any even values; moreover, an explicit formula is obtained for computing the index. The solution of the question of the index is closely connected with the problem of the homotopic classification of elliptic systems of equations, also posed by I. M. Gel' fand ([2]). In particular, with problem (I) there is associated the question of the homotopic classification of first-order systems on the sphere, the solution of which is given by Theorem 2.

1. We consider the following boundary-value problem: to find in the domain D a continuous and twice continuously differentiable solution u of the system $\Delta u = 0$, satisfying the boundary condition

$$\lim_{x\to y} B\left(y, \frac{\partial}{\partial x}\right) u(x) = f(y) \qquad (x \in D,\ y \in S). \tag{I}$$

Here D is a finite convex domain in the space $x = (x^1, x^2, x^3)$, bounded by a triply smooth surface S; Δ is the Laplace operator;

$$B\left(y, \frac{\partial}{\partial x}\right) u(x) = b\frac{\partial u}{\partial \nu} + B_1\left(y, \frac{\partial}{\partial x}\right) u + B_0(y)u,$$

$$B_1\left(y, \frac{\partial}{\partial x}\right) = B^k(y)\frac{\partial}{\partial x^k}, \tag{1}$$

where b is a complex number, $\partial/\partial\nu$ is differentiation in the direction of the normal to S at the point y; $B^k(y)$ $(k = 0, 1, 2, 3)$ are complex square matrices of order p, given and continuous on S, with $B_1(y,\nu)=0$ (ν is the normal vector to S at the point y); $f(y)$ is a given continuous column on S of height p; u is a column of height p.

The following conditions are assumed to be fulfilled: 1) $B^k(y)$ for $k=0$ have first, and for $k=1,2,3$ second, continuous derivatives along S; 2) the eigenvalues of the matrix $B_1(y,i\tau)$, for any $y\in S$ and any unit tangent vectors τ to S at the point y, do not lie on the segment ρb ($0\le\rho\le 1$).

Then from the works of Z. Ya. Shapiro ([3]) and Ya. B. Lopatinskii ([4]) it follows that problem (I) and the adjoint problem (I*) can be reduced to regular integral equations. Hence, as is known, it follows that the homogeneous problems (I) and (I*) have finite numbers (k and k^*, respectively) of linearly independent solutions and that, for the solvability of problem (I), it is necessary and sufficient that $f(y)$ be orthogonal to all solutions of the homogeneous problem (I*).

The derivation of a formula for the index $\varkappa = k-k^*$ is connected with consideration of certain questions relating to systems of equations on a surface, to which Sec. 2 is devoted.

2. Let the surface S be covered by a finite system Ω of open sets, and for each $S'\in\Omega$ let there be given a homeomorphic and three-times continuously differentiable mapping $x=x(\xi)$ of a plane domain onto S', with matrix of first derivatives of rank 2. Then on S a system of equations is defined

$$A^j(\xi)\frac{\partial v}{\partial \xi^j}+A^0(\xi)v=g(\xi)\qquad (\xi=(\xi^1,\xi^2)), \tag{2}$$

so that $B^k(x(\xi))=A^j(\xi)\partial x^k/\partial\xi^j,\quad B^0(x(\xi))=A^0(\xi)$. By condition 2), this system is elliptic. The assertions made in this section are valid for an arbitrary elliptic system of first-order equations on S with sufficiently smooth coefficients in the coordinate systems under consideration on S. It is proved that the number p of equations in system (2) is even, and that the number of roots λ of the polynomial $\delta(\lambda)=\det(A^1(\xi)+\lambda A^2(\xi))$ lying in the upper λ-half-plane is equal to $r=p/2$.

Let, for definiteness, an exterior orientation be chosen on S, and let $x=x(\xi)$ be an arbitrary point of the surface S. Denote by $V(x)$ the space spanned by the columns of the matrix

$$\int_\gamma (A^1(\xi)+\lambda A^2(\xi))^{-1}\,d\lambda,$$

where γ is a contour in the half-plane $\operatorname{Im}\lambda>0$ enclosing all roots of the polynomial $\delta(\lambda)$ that lie in this half-plane. $V(x)$ is an r-dimensional space, independent of the arbitrariness in the choice of the coordinate system on S of one orientation.

In general, if at each point $x \in S$ an r-dimensional subspace $W(x)$ of the p-dimensional complex space of column vectors of height p is given, and if every point of the surface S has a neighborhood in which there is a basis of the space $W(x)$ depending continuously on x, then $I(W)$ will denote the integer defined by the obstruction to constructing a basis continuous on all of S. More precisely, if Γ is an arbitrary curve on S dividing S into two parts S_1 and S_2, homeomorphic to a disk, and if $\omega_1(x)$ and $\omega_2(x)$ are $p \times r$ matrices whose columns form bases of the space $W(x)$ continuous on S_1 and S_2, respectively, then

$$I(W) = \frac{1}{2\pi} \left[\arg\det(\overline{\omega_2'(x)}\, \omega_1(x))\right]_\Gamma,$$

where Γ is oriented coherently with S_1; the prime and the bar denote transposition and complex conjugation. The actual computation of the quantity $I(W)$ can be carried out analogously to the way this is done in (5).

The integer $\chi = I(V)$, uniquely determined by system (2), will be called the characteristic of this system.

Theorem 1. *For any even number p and integer χ, there exists an elliptic system of equations (2) whose characteristic is equal to χ.*

The proof is carried out by an actual construction.

The following theorem on the homotopy classification of systems (2) is proved; as usual, two systems are assigned to the same homotopy class if they can be transformed into one another by a continuous deformation of the coefficients with preservation of the ellipticity condition.

Theorem 2. *In order that two elliptic systems of equations (2) belong to the same homotopy class, it is necessary and sufficient that they have equal characteristics.*

Necessity is obvious. In proving sufficiency, one uses the fact that the space of characteristic matrices $X(\lambda)$ of elliptic systems of first-order equations for which $\det X(\lambda)$ has exactly k roots in the upper λ-half-plane is linearly connec-

but, for $0 < k < p$, its two-dimensional homotopy group is a free cyclic group.

Under the assumption that the right-hand side of system (2) satisfies the Hölder condition and that the solution of this system is sought in the class of functional columns having first continuous derivatives, the following theorem has been proved:

Theorem 3. a) *The homogeneous system (2) ($g = 0$) has a finite number k of linearly independent solutions; b) there exists a finite number l of rows h_j ($j = 1, \dots, l$), continuous on S, of p elements, such that, for the solvability of system (2), it is necessary and sufficient that*

$$\iint h_j g\,dS = 0 \qquad (j = 1, \dots, l);$$

c) the index $\varkappa = k - l$ of system (2) is computed by the formula

$$\varkappa = -2\chi + p,$$

where χ is the characteristic of system (2).

For the proof, on S a special system of coordinates is chosen in which Ω consists of two sets, and a boundary-value problem in a plane domain is constructed for a system of $2p$ equations; between the solutions of this problem and the solutions of system (2) a definite correspondence is established. To the indicated boundary-value problem the results of the paper [6] are applied.

3. **Theorem 4.** *Let $x \in S$; let ν be the vector of the exterior normal to S at the point x; let η^1, η^2 be arbitrary vectors such that ν, η^1, η^2 is a positively oriented frame. Further, let $V(x)$ be the space spanned by the columns of the matrix*

$$\int_\gamma B_1^{-1}(x, \eta^1 + \lambda\eta^2)\,d\lambda,$$

where γ is a contour in the half-plane $\operatorname{Im}\lambda > 0$ enclosing all roots λ of the polynomial $\det B_1(x, \eta^1 + \lambda\eta^2)$ that lie in this half-plane. Then $V(x)$ is a $\frac{1}{2}p$-dimensional space, independent of the arbitrariness in the choice of η^1, η^2. The index of problem (I) is computed by the formula

$$\varkappa = -2I(V) + p. \tag{3}$$

Proof. Consider the problem (I_t), differing from problem (I) only in that in (1), in place of b, there stands tb $(0 \le t \le 1)$. Obviously, for problem (I_t) condition 2) is fulfilled, and therefore its index does not depend on t. For $f = g$, satisfying the Hölder condition on S, between the solutions of problem (I_0) and system (2) a one-to-one correspondence is established: $\Delta u = 0$, $u|_S = v$, whence the coincidence of their indices follows.

Relying on Theorem 1 and formula (3), it is not difficult to show that, for any integer $p > 1$ and any even integer m, there exists a problem (I) whose index is equal to m.

4. From formula (3), obviously, there follows the index formula for the system of two-dimensional singular integral equations to which problem (I) is reduced by means of the simple-layer potential. Relying on this formula, one can obtain a formula for the index of a broader class of systems of

singular integral equations and boundary-value problems, expressing the index in terms of three-dimensional homotopy groups of the unitary group.

Lviv Forestry Engineering
Institute

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Note: Figure translations are in progress. See original paper for figures.