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Abstract

Full Text

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GEOPHYSICS

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A FINITE-DIFFERENCE SCHEME FOR SOLVING THE COMPLETE SYSTEM OF EQUATIONS OF SHORT-RANGE WEATHER FORECASTING AND QUASI-GEOSTROPHIC RELATIONS

The problem of short-range weather forecasting by means of the complete system of equations of hydrodynamics leads to the determination of four functions: the geopotential Φ, the horizontal wind components u, v, and the quantity $\bar{\omega}$, equivalent to the vertical component of velocity: $\bar{\omega} = \frac{1}{P}\frac{dp}{dt}$ (p is pressure, P is the standard pressure at sea level, t is time). To determine these functions one may use the four equations

$$\frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x} - lv = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \bar{\omega}\frac{\partial u}{\partial \zeta}; \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial \Phi}{\partial y} + lu = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \bar{\omega}\frac{\partial v}{\partial \zeta}; \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \bar{\omega}}{\partial \zeta} = 0; \tag{3}$$

$$\zeta^2 \frac{\partial^2 \Phi}{\partial \zeta\, \partial t} + c^2 \bar{\omega} = -\zeta^2 \left(u\frac{\partial^2 \Phi}{\partial x\, \partial \zeta} + v\frac{\partial^2 \Phi}{\partial y\, \partial \zeta} \right). \tag{4}$$

Here the independent variables are: x, y—horizontal coordinates (the Y-axis is directed northward) and the reduced pressure $\zeta = p/P$; l is the Coriolis parameter (depends on y); $c^2 = \alpha R T_1$; R is the gas constant; T_1 is the mean temperature; $\alpha = (\gamma_a - \gamma)R/g$ (g is the acceleration due to gravity, γ is the mean vertical temperature gradient, γ_a is the adiabatic (or pseudoadiabatic) gradient). In this case the vertical velocity w is determined from the relation

$$-gw = \frac{RT_1}{\zeta}\bar{\omega} - \frac{\partial\Phi}{\partial t} - u\frac{\partial\Phi}{\partial x} - v\frac{\partial\Phi}{\partial y}, \tag{5}$$

and the temperature T and density ρ are found from the formulas

$$T = -\frac{\zeta}{R}\frac{\partial\Phi}{\partial\zeta}, \qquad \frac{1}{\rho} = -\frac{1}{P}\frac{\partial\Phi}{\partial\zeta}. \tag{6}$$

The boundary conditions adopted are

$$\zeta w = 0 \quad \text{for } \zeta = 0; \tag{7}$$

$$w = 0 \quad \text{for } \zeta = 1 \tag{8}$$

(the case of absence of mountains is considered). The medium is assumed infinitely extended in the directions of the X and Y axes (the formulation of the short-range forecasting problem), and the curvature of the Earth is taken into account only by specifying l.

as functions of y. In the course of solving the problem, boundedness conditions are imposed.

The system (1)–(4) is of third order in time; it is sufficient to prescribe u, v, Φ at the initial instant. In this case $\bar{\omega}$ is determined from (3) and (7) in the form

$$\bar{\omega} = -\int_0^{\zeta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)d\xi. \tag{9}$$

Eliminating $\bar{\omega}$ from (4) by means of (3), we arrive at a system of three equations

$$\frac{\partial u}{\partial t} + \frac{\partial\Phi}{\partial x} - lv = -B_u; \tag{10}$$

$$\frac{\partial v}{\partial t} + \frac{\partial\Phi}{\partial y} + lu = -B_v; \tag{11}$$

$$\frac{\partial}{\partial\zeta}\left(\zeta^2\frac{\partial^2\Phi}{\partial\zeta\,\partial t}\right) - c^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = R\frac{\partial\zeta A_T}{\partial\zeta}, \tag{12}$$

where

$$B_u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \bar{\omega}\frac{\partial u}{\partial\zeta}, \qquad B_v = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \bar{\omega}\frac{\partial v}{\partial\zeta}; \tag{13}$$

$$A_T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}.$$

$\bar{\omega}$ and T are determined through u, v, Φ by (6) and (9).

The system (10), (11), (12) is solved with the boundary condition

$$\frac{\partial^2 \Phi}{\partial \zeta\, \partial t} + \alpha \frac{\partial \Phi}{\partial t} = RA_T \quad \text{for } \zeta = 1. \tag{14}$$

Let us approximately replace the time derivatives by finite-difference relations

$$\frac{\partial u}{\partial t} \approx \frac{u - u^0}{\delta t}, \qquad \frac{\partial v}{\partial t} \approx \frac{v - v^0}{\delta t}, \qquad \frac{\partial \Phi}{\partial t} \approx \frac{\Phi - \Phi^0}{\delta t}$$

(u^0, v^0, Φ^0 are the values of the corresponding functions at the initial instant of time).

The system (10), (11), (12) is replaced by the finite-difference scheme

$$u + \delta t\left(\frac{\partial \Phi}{\partial x} - lv\right) = -\delta t \cdot B_u + u^0; \tag{15}$$

$$v + \delta t\left(\frac{\partial \Phi}{\partial y} + lu\right) = -\delta t \cdot B_v + v^0; \tag{16}$$

$$\frac{\partial}{\partial \zeta}\left(\zeta^2 \frac{\partial \Phi - \Phi^0}{\partial \zeta}\right) - c^2 \delta t \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = R\frac{\partial \zeta A_T}{\partial \zeta}, \tag{17}$$

where the boundary condition (14) takes the form

$$\frac{\partial}{\partial \zeta}(\Phi - \Phi^0) + \alpha(\Phi - \Phi^0) = R\delta t \cdot A_T \quad \text{for } \zeta = 1. \tag{18}$$

Eliminating the velocities u, v from (17) by means of (15), (16) and introducing the dimensionless parameter $l_0 \delta t = \delta$ (l_0 is the value of l at the origin of coordinates), we obtain···

we arrive at the equation

$$\left[\frac{1+\delta^2}{\delta^2}\frac{\partial}{\partial \zeta}\zeta^2\frac{\partial}{\partial \zeta} + \frac{c^2}{l_0^2}\left(\Delta + b\delta\frac{\partial}{\partial x} - b'\frac{\partial}{\partial y}\right)\right](\Phi - \Phi^0) =$$
$$= \delta\left[\frac{R(1+\delta^2)}{l_0\delta^2}\frac{\partial}{\partial \zeta}(\zeta A_T) - \frac{c^2}{l_0^2}\left(\frac{\partial B_v}{\partial x} - \frac{\partial B_u}{\partial y}\right) + b\frac{\partial \Phi^0}{\partial x}\right] -$$

$$-\frac{c^2}{l_0^2}\left[\frac{\partial B_u}{\partial x}+\frac{\partial B_v}{\partial y}+\Delta\Phi^0-l_0\left(\frac{\partial v^0}{\partial x}-\frac{\partial u^0}{\partial y}\right)-b'\frac{\partial \Phi^0}{\partial y}\right]+\frac{c^2}{l_0\delta}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}\right), \tag{19}$$

where Δ is the plane Laplace operator,

$$b=\left(\frac{1}{l}\frac{dl}{dy}\right)_0\frac{\delta^2-1}{\delta^2+1}, \qquad b'=2\left(\frac{1}{l}\frac{dl}{dy}\right)_0\frac{\delta^2}{\delta^2+1}. \tag{20}$$

Taking the right-hand sides as known, we find Φ, integrating (19) under condition (18), in the form

$$\Phi=\delta\Phi_I+\Phi_{\text{II}}+\frac{1}{\delta}\Phi_{\text{III}}, \tag{21}$$

where

$$\Phi_I=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^1\left[G\left(\frac{\partial B_v}{\partial x'}-\frac{\partial B_u}{\partial y'}+b\frac{\partial \Phi^0}{\partial x'}\right)+\frac{Rl_0}{c^2}\frac{1+\delta^2}{\delta^2}\zeta'\frac{\partial G}{\partial \zeta'}A_T\right]d\zeta'\,dx'\,dy'; \tag{22}$$

$$\Phi_{\text{II}}=\Phi^0+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^1 G\left[\frac{\partial B_u}{\partial x'}+\frac{\partial B_v}{\partial y'}+\Delta\Phi^0-l_0\left(\frac{\partial v^0}{\partial x'}-\frac{\partial u^0}{\partial y'}\right)-b'\frac{\partial \Phi^0}{\partial y'}\right]d\zeta'\,dx'\,dy'; \tag{23}$$

$$\Phi_{\text{III}}=-\frac{l_0}{2\pi}\int_{-\infty}^{+\infty}\int_0^1 G\left(\frac{\partial u^0}{\partial x'}+\frac{\partial v^0}{\partial y'}\right)d\zeta'\,dx'\,dy'; \tag{24}$$

$$G=\frac{1}{2\sqrt{\zeta\zeta'}}\left[S\left(\ln\frac{\zeta'}{\zeta},r\right)+S\left(\ln\frac{1}{\zeta\zeta'},r\right)+\right.$$
$$\left.+(1-2\alpha)(\zeta\zeta')^{1/2-\alpha}\int_{\infty}^{\ln\frac{1}{\zeta\zeta'}}e^{(1/2-\alpha)\xi}S(\xi,r)\,d\xi\right]\exp\left[-b\delta\frac{x-x'}{2}+b'\frac{y-y'}{2}\right]; \tag{25}$$

$$S(\xi,r)=\left(\xi^2+\frac{1+\delta^2}{\delta^2}\frac{l_0^2}{c^2}r^2\right)^{-1/2}\exp\left[-\sqrt{\frac{1+\alpha^2}{2}}\sqrt{\xi^2+\frac{1+\delta^2}{\delta^2}\frac{l_0^2}{c^2}r^2}\right];$$

$$r^2=(x-x')^2+(y-y')^2, \qquad \alpha^2=\frac{c^2}{l_0^2}\left(\frac{1}{l}\frac{dl}{dy}\right)_0^2\frac{\delta^4}{1+\delta^2}.$$

In this case u, v, $\bar{\omega}$ can be found from (15), (16), and (4) in the form

$$u = \frac{\delta^2}{1+\delta^2}\left(\delta u_I + u_{\text{II}} + \frac{1}{\delta}u_{\text{III}} + \frac{1}{\delta^2}u_{\text{IV}}\right); \tag{26}$$

$$v = \frac{\delta^2}{1+\delta^2}\left(\delta v_I + v_{\text{II}} + \frac{1}{\delta}v_{\text{III}} + \frac{1}{\delta^2}v_{\text{IV}}\right); \tag{27}$$

$$\bar{\omega} = \bar{\omega}_{\text{II}} + \frac{1}{\delta}\bar{\omega}_{\text{III}} + \frac{1}{\delta^2}\bar{\omega}_{\text{IV}}, \tag{28}$$

where

$$u_I = -\frac{1}{l_0}\frac{\partial\Phi_I}{\partial y}; \qquad v_I = \frac{1}{l_0}\frac{\partial\Phi_I}{\partial x};$$

$$u_{\text{II}} = -\frac{1}{l_0}\frac{\partial\Phi_{\text{II}}}{\partial y} - \frac{B_v}{l_0} - \frac{1}{l_0}\frac{\partial\Phi_{\text{I}}}{\partial x}; \qquad v_{\text{II}} = \frac{1}{l_0}\frac{\partial\Phi_{\text{II}}}{\partial x} + \frac{B_u}{l_0} - \frac{1}{l_0}\frac{\partial\Phi_{\text{I}}}{\partial y};$$

$$u_{\text{III}} = -\frac{1}{l_0}\frac{\partial\Phi_{\text{III}}}{\partial y} - \frac{1}{l_0}B_u + v^0 - \frac{1}{l_0}\frac{\partial\Phi_{\text{III}}}{\partial x}; \tag{29}$$

$$v_{\text{III}} = \frac{1}{l_0}\frac{\partial\Phi_{\text{III}}}{\partial x} - \frac{B_v}{l_0} - u^0 - \frac{1}{l_0}\frac{\partial\Phi_{\text{II}}}{\partial y};$$

$$u_{\text{IV}} = u^0 - \frac{1}{l_0}\frac{\partial\Phi_{\text{III}}}{\partial x}; \qquad v_{\text{IV}} = v^0 - \frac{1}{l_0}\frac{\partial\Phi_{\text{III}}}{\partial y};$$

$$\overline{\omega}_{\text{II}} = \frac{l_0\zeta}{c^2}\left(RA_T - \zeta\frac{\partial\Phi_{\text{I}}}{\partial\zeta}\right); \qquad \overline{\omega}_{\text{III}} = \frac{l_0\zeta^2}{c^2}\frac{\partial\Phi^0 - \Phi_{\text{II}}}{\partial\zeta}; \qquad \overline{\omega}_{\text{IV}} = -\frac{l_0\zeta^2}{c^2}\frac{\partial\Phi_{\text{III}}}{\partial\zeta}.$$

The first of the three terms on the right-hand side of (21) (containing the factor δ) may conventionally be called the **evolutionary** term, the second the **stationary** term, and the third the **damping** term (we assume $\delta \gg 1$, which means that $\delta t \gg 2$ h 30 min). Analogous terms are contained in the right-hand sides of u, v; ω does not contain an evolutionary term. Let us note that, completely discarding the nonlinear terms (B_u, B_v, A_T) and neglecting $\frac{dl}{dy}$ $(b = b' = 0)$, we obtain $\Phi_{\text{I}} = u_{\text{I}} = v_{\text{I}} = \overline{\omega}_{\text{II}} = 0$. Φ, u, v will contain only stationary terms (which will be connected by the condition of geostrophy) and damping terms. This result is in agreement with ([1,2]). This solution has no prognostic meaning.

Discarding B_u, B_v, A_T, but retaining the terms with b and b', we obtain evolution $(\Phi_{\text{I}} \neq 0)$. A more complete solution, which can already be applied to forecasting, corresponds to a linearization of B_u, B_v, A_T with respect to some basic motion. Finally, the general case of forecasting is obtained by retaining the nonlinear terms in full.

It is remarkable that, in the most general case, the principal (evolutionary) terms for Φ, u, v will always be connected, according to (21), (29), by the conditions of geostrophy.

The time steps used up to now in quasigeostrophic short-range forecasting ([3]) are obtained from our formula (21) if in it only the evolutionary term is retained, while B, B_v, A_T are replaced by B_u^0, B_v^0, A_T^0, substituting for u, v their evolutionary parts (and zero instead of $\overline{\omega}$), setting in (25) $b = b' = a = 0$, and replacing $\frac{\delta^2}{1+\delta^2}$ by 1.

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Note: Figure translations are in progress. See original paper for figures.