MATHEMATICS

1960

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Abstract

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MATHEMATICS

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ON THE PROBLEM OF ISOMORPHISM OF STRUCTURES

(Presented by Academician A. I. Mal̆ cev on 12 I 1960)

A structure specified by a finite set of generating elements and defining relations will be called **finitely defined**. In the present article the isomorphism problem is solved positively for finitely defined structures. In doing so, we use the connection between finitely defined structures and free extensions of finite structuroids ([1]).

We shall denote by $P(x_1, x_2, \dots, x_n; S)$ the finite structuroid P with elements $x_1, x_2, \dots, x_n$ and Cayley table S for intersections and unions of its elements. From the theorem on the embeddability of structuroids in structures ([1]) and from ([3]) it follows that the free extension of the structuroid $P(x_1, x_2, \dots, x_n; S)$ is a structure specified by generating elements $x_1, x_2, \dots, x_n$ and a system of defining relations S; we shall denote it by $FL(P)$.

Conversely, if L is a finitely defined structure, then it is the free extension of some finite structuroid, for the determination of which an algorithm is indicated by Evans ([3]); we shall call it the **Evans algorithm**.

The operations $\cap$ and $\cup$, partially defined in the structuroid $P(x_1, x_2, \dots, x_n; S)$, may already be defined by some subsystem S' of S, i.e. the Evans algorithm, applied to the elements $x_1, x_2, \dots, x_n$ and to the system of defining relations S', leads to $P(x_1, x_2, \dots, x_n; S)$. In this case we shall say that the relations of the system $S \setminus S'$ are consequences of the relations of the system S' and of the axioms of the structuroid.

Definition 1. A system of defining relations of the structuroid $P(x_1, x_2, \dots, x_n; S)$ will be called **tabular** if each relation in it has the form $x_i * x_j = x_k$, where $*$ denotes $\cap$ or $\cup$.

Definition 2. A tabular system of defining relations of a structuroid will be called **irreducible** if none of its relations is a consequence of the remaining relations and the axioms of the structuroid.

Using the Evans algorithm, from any tabular system of defining relations of a finite structuroid one can obtain its irreducible system of defining relations and its Cayley table.

Using the Cayley table S of the structuroid $P(x_1, x_2, \dots, x_n; S)$, one can describe the process of constructing $FL(P)$. For this purpose we agree to write unions of elements above, and intersections below, the main diagonal of the Cayley table, and the elements $x_1, x_2, \dots, x_n$ in its entrance row and column in increasing order of indices.

Let $P(x_1, x_2, \dots, x_n; S)$ be a finite structuroid. If all cells of the table S are filled, then P is a structure, and $FL(P) \cong P$. If some cell, for example (i, j), is empty, then we add to P a new element x_{n+1} and to S the relation $x_i \cap x_j = x_{n+1}$, if $i > j$, and $x_i \cup x_j = x_{n+1}$, if $i < j$. As a result we obtain the structuroid $P_1(x_1, x_2, \dots, x_n, x_{n+1}; S_1)$.

If all cells of the table S_1 turn out to be filled, then we shall consider the process completed. If, however, an empty cell is again found in S_1, then analogously we construct P_2. Continuing this process until we reach a structuroid all cells of whose Cayley table are filled,

we obtain a finite or countable sequence of structuroids, each of which is identically embedded in the next. If the identical embeddability of a structuroid P' in a structuroid P'' is denoted by $P' \Rightarrow P''$, then the sequence obtained is written in the form

$$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_k \Rightarrow \cdots \qquad (1)$$

We shall call the sequence (1) a **sequence of extensions of the structuroid** P.

Theorem 1. If the structuroid P_k belongs to some sequence of extensions of the structuroid P, then $FL(P_k) \cong FL(P)$.

From (2) it follows immediately that $FL(P_1) \cong FL(P)$ and $FL(P_k) \cong FL(P_{k-1})$. Consequently, by induction on k the theorem is proved.

Corollary. If the sequence of extensions (1) of the structuroid P is finite and terminates at P_k, then $FL(P) \cong P_k$; if, however, it is infinite, then

$$FL(P) \cong \bigcup_{i=1}^{\infty} P_i.$$

(Here $\cup$ denotes the set-theoretic union of structuroids with preservation of the relations among their elements.)

Definition 3. A structuroid Q will be called a **finitely free extension** of a finite structuroid P if Q belongs to some sequence of extensions of the structuroid P.

Definition 4. A structuroid P_0 will be called a **base of the structuroid** P if P is a finitely free extension of P_0, and P_0 is not a finitely free extension of any structuroid distinct from itself.

Theorem 2. If

$$P'(x_1, x_2, \dots, x_{m-1}; S') \Rightarrow P''(x_1, x_2, \dots, x_m; S''),$$

then, in order that the structuroid P'' be a finitely free extension of the structuroid P', it is necessary and sufficient that every irreducible system of defining relations of the structuroid P'', including the system of defining relations of the structuroid P', contain a relation of the form $x_i * x_j = x_m$ with $x_i \neq x_m$, $x_j \neq x_m$, and contain no other relation involving x_m.

Let $P''(x_1, x_2, \dots, x_m; S'')$ be a finitely free extension of the structuroid $P'(x_1, x_2, \dots, x_{m-1}; S')$, i.e. P' and P'' are neighboring members of a sequence of extensions of some structuroid P. Since P'' does not coincide with P', in the Cayley table S' there are empty cells; let these be

$$(i_1, j_1), \quad (i_2, j_2), \dots, (i_k, j_k) \tag{2}$$

and let P'' be obtained from P' by adjoining the element x_m and a certain relation $x_{i_1} * x_{j_1} = x_m$. In the Cayley table S', together with the cell (i_1, j_1), some other cells from (2) may turn out to be filled by the element x_m. Let these be the cells

$$(i_1, j_1), \quad (i_2, j_2), \dots, (i_r, j_r). \tag{3}$$

Then in S'' the element x_m enters into the relations

$$x_{i_1} * x_{j_1} = x_m, \quad x_{i_2} * x_{j_2} = x_m, \dots, \quad x_{i_r} * x_{j_r} = x_m. \tag{4}$$

In addition, in S'' the element x_m enters into relations of the form

$$x_p \cup x_m = x_q, \quad x_m \cap x_s = x_t. \tag{5}$$

All relations (4) are equivalent to one another in the sense that, after adjoining one of them to S', the remaining ones can be obtained as consequences. Hence, in an irreducible system of defining relations of the structuroid P'', it is sufficient to have only one of the relations (4). It is not difficult to show that none of the relations (4) is a consequence of the relations S' and (5). It follows that in any irreducible system of defining relations R of the structuroid P'' there exists a relation of the form $x_i * x_j = x_m$, with $x_i \neq x_m$, $x_j \neq x_m$, and moreover x_m does not enter into any other relations from R.

The converse assertion is obvious.

Thus the theorem is proved.

Theorem 3. There exists an algorithm that makes it possible, in a finite number of steps, to find a base of a finite structuroid $P(x_1, x_2, \dots, x_n; S)$.

The desired algorithm is as follows: remove from P the element x_i and all relations in which x_i occurs, and determine, using Theorem 2, whether P is a finitely free extension of the newly obtained structuroid

$$P^{(i)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; S^{(i)}).$$

If it turns out that P is not a finitely free extension of the structuroid $P^{(i)}$ for $i = 1, 2, \dots, n$, then P itself is its own base. If, however, for some i, P is a finitely free extension of $P^{(i)}$, then we carry out the same process as applied to the structuroid $P^{(i)}$. Since in doing so the number of elements of the structuroid has decreased by one, and the number of elements in P is finite, we shall ultimately arrive at a structuroid P_0 that is a base of P.

Definition 5. An element x_i of the structuroid $P(x_1, x_2, \dots, x_n; S)$ will be called **k-removable** if P is a finitely free extension of some $(n-k)$-element structuroid that does not contain the element x_i, and is not a finitely free extension of any $(n-k+1)$-element structuroid that does not contain the element x_i.

We shall call an element x_i **removable** if it is k-removable for some k; in this case k will be called the **order of removability** of the element x_i.

By induction on the number of elements of the structuroid it is easy to prove that 1-removable elements of a structuroid cannot belong to its base; after this, by induction on the order of removability, one proves:

Theorem 4. A removable element of a finite structuroid cannot belong to its base.

Theorem 5. Every finite structuroid has a unique base.

Let P have bases P_0 and Q. If $x \in P_0$ and $x \notin Q$, then x is a removable element of the structuroid P, and consequently, by Theorem 4, $x \notin P_0$. Hence from $x \in P_0$ it follows that $x \in Q$, which, in view of the equality of rank of P_0 and Q, gives $P_0 = Q$.

Theorem 6. Let P' and P'' be two finite structuroids. In order that the structures $FL(P')$ and $FL(P'')$ be isomorphic, it is necessary and sufficient that the bases of the structuroids P' and P'' be isomorphic.

Sufficiency follows from the uniqueness of the free extension ([2]), and necessity from the uniqueness of the base of a finite structuroid.

The algorithm that makes it possible to decide the question of isomorphism of two finitely defined structures L_1 and L_2 can now be described as follows.

1. Applying Evans' algorithm, we find structuroids P' and P'' such that

$$FL(P') \cong L_1$$

and

$$FL(P'') \cong L_2.$$

2. We find the bases P_0', P_0'' of the structuroids P' and P'', respectively.
3. We investigate the question of isomorphism of the structuroids P_0' and P_0''. Owing to the finiteness of the structuroids P_0' and P_0'', there exists an algorithm that decides the question of their isomorphism.

Starting from the uniqueness of the base of a finite structuroid P and from the uniqueness of its free extension $FL(P)$, it is easy to prove that the automorphism group of the structure $FL(P)$ is isomorphic to the automorphism group of the base P_0 of the structuroid P. Hence, and from the fact that every finitely defined structure L is isomorphic to some structure $FL(P)$, it follows that there exists an algorithm for finding the automorphism group of an arbitrary finitely defined structure L. This group will be isomorphic to some subgroup of the symmetric group S_n, where n is the number of elements of the structuroid P_0 that is the base of the structuroid P for which $FL(P) \cong L$.

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Received
9 I 1960

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Note: Figure translations are in progress. See original paper for figures.