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Abstract

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ON THE EXISTENCE OF A WEAK SOLUTION OF THE DIRECT PROBLEM IN THE THEORY OF FLOW PAST A PROFILE BY A SONIC STREAM IN THE FIRST APPROXIMATION

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By the hodograph method the author constructed an example of symmetric flow past a symmetric profile by a steady stream with the speed of sound at infinity ([1]). The example was constructed both for Tricomi gas and for Chaplygin gas. The problem solved has the character of an **inverse problem**, since the profile being flowed past is found a posteriori.

The first **direct problem** of flow past a profile by a sonic stream was solved in independent works by L. V. Ovsyannikov ([2]) and Guderley–Yoshihara ([3]): this is the problem of flow past a symmetric quadrangular profile for Tricomi gas. It turned out that this problem reduces to the Tricomi problem and, consequently, is solved uniquely.

In the present work the problem of flow past a smooth profile is solved by the hodograph method, analogously to the way in which we solved the problem of transonic flow in a Laval nozzle with smooth walls ([4,5]). A case close to that of symmetric flow past a symmetric profile is considered, and the problem, by linearizing the boundary conditions, is solved in the first approximation. The existence of a weak solution of the problem is proved in the sense in which this term is used in ([4]).

The uniqueness of the solution of the problem, just as in the problem of a nozzle with smooth walls, has not yet been proved in this first approximation: one undetermined parameter enters the solution. The question of the possibility of determining this parameter uniquely in a rigorous solution of the problem remains open for the time being. Contrary to the hypothesis advanced by the author in ([4,5]), some theoretical and experimental results speak in favor of uniqueness both in the case of transonic flow in a nozzle ([6]), and in the case of flow past a smooth profile by a supersonic stream with a detached shock wave ([7]), which, as $M_\infty \to 1$, passes into the case considered by us. A discussion of this question is given at the end of the article.

The stream function of the example constructed by the author ([1]) corresponds to a rounded leading edge. Denote this stream function by Ψ. The profile $\bar{L}$ being flowed past is given by the equation

$$\Psi = 0, \tag{1}$$

where for the upper and lower sides of the profile one must take, respectively, those branches of this curve which at the point $u = v = 0$ of the hodograph plane are tangent to the half-lines $\theta = \pm\pi/2$.

The function Ψ in a neighborhood of the origin of coordinates of the plane is represented in the form

$$\Psi = \rho^{-1/3} f_0\left(\frac{\theta}{\rho}\right) + \rho^{-1} f_1\left(\frac{\theta}{\rho}\right) + \rho^{-1/3} f_2\left(\frac{\theta}{\rho}\right) + \rho^{1/3} f_3\left(\frac{\theta}{\rho}\right) + \delta\Psi, \tag{2}$$

where

$$\rho = \sqrt{\theta^2 + \frac{4}{9}\eta^3}, \qquad f_0\left(\frac{\theta}{\rho}\right) = \left(1 - \frac{\theta}{\rho}\right)^{1/3}\left(\frac{1}{3} + \frac{\theta}{\rho}\right) - \left(1 + \frac{\theta}{\rho}\right)^{1/3}\left(\frac{1}{3} - \frac{\theta}{\rho}\right). \tag{2a}$$

The first four terms on the right-hand side are regular on the characteristics $\rho^2 = 0$, with the exception of the origin itself. The remainder term $\delta\Psi$, together with its derivatives with respect to the characteristic coordinates $\partial\delta\Psi/\partial\lambda$, $\partial\delta\Psi/\partial\mu$ ($\lambda = \theta - \frac{2}{3}(-\eta)^{3/2}$, $\mu = \theta + \frac{2}{3}(-\eta)^{3/2}$), remains finite for $\rho^2 = 0$ on these characteristics, including the origin.

At present it has not yet been verified whether the line $\overline{L}$ in the region $\eta < 0$, $\rho^2 \geqslant 0$ satisfies the inequality

$$-(-\eta)^{-1/2} < \frac{d\eta}{d\theta} < (-\eta)^{-1/2}. \tag{3}$$

If this is not so, then instead of $\overline{L}$ we prescribe another curve L_0, symmetric with respect to the θ-axis and satisfying this condition, with $\theta \to \pm\pi/2$ as $\eta \to +\infty$, and replace Ψ by another solution:

$$\psi_0 = \Psi + \delta\psi_0, \tag{4}$$

where $\delta\psi_0$ is a solution, in the worst case weak* in the sense of K. S. Morawetz ([8]), of S. A. Chaplygin' s equation

$$K\delta\psi_{0\theta\theta} + \delta\psi_{0\sigma\sigma} = 0 \tag{5}$$

with boundary condition

$$\delta\psi_0 = -\Psi \quad \text{on } L_0, \tag{6}$$

where the quantities corresponding to the derivatives $\delta\psi_{0\theta}$, $\delta\psi_{0\sigma}$ must belong to a certain generalized Hilbert space.

Let us apply a hodograph transformation to the stream function Ψ, i.e. let us find

$$\Omega = \Psi - \frac{\sigma}{\sigma_0}(uY - vX), \tag{7}$$

where $u = w\cos\theta$; $v = w\sin\theta$; $\sigma = \sigma_0\left(1 - \frac{w^2}{w_m^2}\right)^{\frac{1}{\chi-1}}$ is the density**, corresponding to the velocity w; X, Y are Cartesian coordinates in the physical plane corresponding to the stream function Ψ.

In a neighborhood of the origin of the (θ, η)-plane we have

$$\Omega = -\left(\frac{3}{2}\right)^{1/3}(\chi+1)^{1/3}\rho^{-1/3}\left[\left(1+\frac{\theta}{\rho}\right)^{2/3} - \left(1-\frac{\theta}{\rho}\right)^{2/3}\right] + o(1). \tag{8}$$

We now pass to another, arbitrary profile L, which is displaced from the profile L_0 in the direction of the normal (taking the positive direction to be from right to left when looking downstream) by a small distance $\delta n(s)$, where s is an arbitrary parameter varying along the profile L_0; at the same time the varied profile L, like L_0, must have no corner points. Then the boundary condition for the variation*** $\delta\omega = \omega - \omega_0$ on L_0 is (see [10,11])

$$\delta\omega = -\frac{\sigma}{\sigma_0} w \delta n \tag{9}$$

(up to an error of second order of smallness).

* The question of when such weak solutions are strong and when they are strict was partially investigated in the work of K. O. Friedrichs [9].

** We use here the letter σ for density instead of the commonly used p, since in our notation ρ denotes $\sqrt{\theta^2 + \frac{4}{9}\eta^3}$.

*** The functions ω, ω_0 are obtained from ψ, ψ_0 in the same way as Ω from Ψ (see formula (7)).

The function $\delta\omega$ must have the form

$$\delta\omega = \varepsilon\Omega + \delta\omega', \tag{10}$$

where the function $\delta\omega'$ is defined in the region of the (θ, η)-plane bounded by the curve L_0 and by the characteristics $\rho^2 = 0$ (see Fig. 1).

Thus, for $\delta\omega'$ we have the boundary condition

$$\delta\omega' = -\frac{\sigma}{\rho_0}\,\omega\delta n - \varepsilon\Omega. \tag{11}$$

The function $\delta\omega'$ satisfies the equation

$$\delta\omega'_{\theta\theta} + \frac{\partial}{\partial\hat{\sigma}}\left(\frac{1}{\hat{K}}\frac{\partial\delta\omega'}{\partial\hat{\sigma}}\right) = 0, \tag{12}$$

where $\hat{\sigma}, \hat{K}$ are functions of the velocity modulus introduced by the author in paper [4].

The boundary-value problem (11) for equation (12), as shown in paper [4], has, at least, a weak solution.* As K. S. Morawetz [12] showed, this solution is the unique solution in the proper sense of the word if it is a strict solution (with some additional conditions at singular points).

However, we have seen that our solution

$$\omega = \omega_0 + \varepsilon\Omega + \delta\omega'(\theta, \eta; \varepsilon) \tag{13}$$

contains the undetermined small parameter ε.

Let us dwell briefly once more on the problem of flow past a smooth profile with a detached shock wave, qualitatively close to the problem considered above. In formulating this boundary-value problem in the hodograph plane, the whole change in comparison with the preceding problem consists in the fact that, instead of the character of the singularity at the origin of coordinates of the (θ, η)-plane, one must prescribe conditions on the hodograph of the bow wave.

Fig. 1

For the initial flow with stream function ψ_0 we have (see [13]):

$$d\varphi_0 = \frac{\sigma_0}{\sigma_\infty}\operatorname{ctg}\gamma\, d\psi_0, \tag{14}$$

where γ is the local angle of inclination of the bow wave and σ_∞ is the density in the incident flow. For the variation of the transformed stream function $\delta\omega$ we then have

$$d\delta\varphi = \frac{\sigma_0}{\sigma_\infty}\operatorname{ctg}\gamma\, d\delta\psi, \tag{15}$$

where

$$\delta\psi = \delta\omega - \frac{w\delta\omega_w}{1-M^2}, \tag{16}$$

$$\delta\varphi = \frac{\sigma_0}{\sigma}\delta\omega_\theta + \int \left(\frac{\sigma_0 w}{\sigma(1-M^2)}\,\delta\omega_w\,d\theta - \frac{\sigma_0}{\sigma w}\,\delta\omega_\theta\,d\omega \right). \tag{17}$$

The boundary condition (15) is, with respect to $\delta\omega$, a second-order differential relation. Therefore one should expect that the solution depends on two parameters. The solution of the problem could prove to be uni—

* In these solutions the quantities corresponding to the derivatives $\delta\omega'_\theta$, $\delta\omega'_{\hat{\sigma}}$ must belong to some prescribed generalized Hilbert space.

···nonidentical if the requirement

$$\lim_{w\to a^*} \frac{1}{1-M^2}\frac{\partial\delta\omega}{\partial w} \neq \infty \tag{18}$$

at the points of intersection of the line L_0 with the sonic line were to lead to two nonidentical and independent equations for determining the indicated parameters*. This is possible, since from (9), (18) it follows that

$$\lim_{w\to a^*} \frac{\partial\delta\omega}{\partial\theta} = -\lim_{w\to a^*} \frac{d}{d\theta}\left(\frac{\sigma}{\sigma_0}w\delta n\right), \tag{19}$$

where the total derivative on the right-hand side is taken along the line L_0.

If condition (18) is replaced by the stronger condition (18a), requiring boundedness and continuity of all second derivatives of the function $\delta\omega$ in a neighborhood of the indicated points, then at them, in addition to (19), we obtain the equation

$$\frac{\partial^2\delta\omega}{\partial\theta^2} + \frac{\partial^2\delta\omega}{\partial w^2}\left(\frac{dw}{d\theta}\right)^2 = -\frac{d^2}{d\theta^2}\left(\frac{\sigma}{\sigma_0}w\delta n\right). \tag{19a}$$

But in the case of the flow around a smooth profile at the speed of sound, we have at our disposal only one parameter ε. Therefore the indicated two equations could be satisfied not at an arbitrary angle of attack (for example, for a symmetric profile at zero angle of attack, but not at small angles of attack different from zero).

In the theory of the Laval nozzle, conditions (18) would have to be satisfied not only at the points of intersection of the sonic line with the hodographs of the walls of the uncurved nozzle, but also at the center of the flow, i.e., we would have 3 equations, whereas in this case we have at our disposal only 2 parameters (the discharge and the angle of inclination of the velocity at the center of the flow). This would mean that a transonic stationary and shock-free flow is not always possible, but apparently is unique in those cases when it is possible, for example in the case of a symmetric nozzle.

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* In the approximate theory of O. M. Belotserkovskii ([7]), uniqueness of the solution is obtained thanks to conditions precisely at these points.

Note: Figure translations are in progress. See original paper for figures.