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Abstract

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MATHEMATICS

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ON THE ADAMS SPECTRAL SEQUENCE

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The purpose of this note is to investigate the structure of the limiting term of the spectral sequence that was introduced by Adams in ([1]). In what follows this spectral sequence will simply be called the Adams sequence. Let us note at once that Adams' assertion on the structure of the term $E_\infty(S^0)$ (S^0 is the zero-dimensional sphere) will follow directly from our general result.

Let X be a space and let S be the suspension operator. We put

$$\pi_m^S(X) = \mathrm{Dir}\lim_{n\to\infty}[\pi_{m+n}(S^nX)]; \qquad \pi_*^S(X) = \sum_{m\geq 0}\pi_m^S(X);$$

$K_m^S(X)$ is the subgroup of the group $\pi_m^S(X)$ consisting of elements of order $q \neq p$, where p is a fixed prime; $K_m(X)$ is the subgroup of the group $\pi_m(X)$ consisting of elements of order $q \neq p$;

$$K_*^S(X) = \sum_{m\geq 0} K_m^S(X); \qquad K_*(X) = \sum_{m\geq 0} K_m(X).$$

We shall denote the Steenrod algebra modulo p, graded according to the degree of operations, by

$$A = \sum_{q\geq 0} A_q,$$

and all groups of the form $H^*(X; Z_p)$ will be regarded as A-modules. The field Z_p can also be made into an A-module if one sets $A_q \cdot Z_p = 0,\ q > 0$, and, since $A_0 \cong Z_p,\ A_0 \cdot Z_p = Z_p \cdot Z_p$. Therefore one can define the groups

$$\mathrm{Ext}_A^{s,t}(H^*(x; Z_p); Z_p).$$

We can now formulate the main theorem of ([1]).

Theorem 1 (Adams). Let X be a space such that the groups $H^t(X; Z_p)$ have finite type for all t. Then there exists a spectral sequence $\{E_r^{s,t}(X), d_r\}$ satisfying the following conditions:

1) $E_r^{s,t}(X) = 0$, if $s < 0$ or $t < s$;

2) $d_r : E_r^{s,t}(X) \to E_r^{s+r,t+r-1}$;

3) $E_2^{s,t}(X) \cong \mathrm{Ext}_A^{s,t}(H^*(X; Z_p), Z_p)$;

4) there exists a canonical monomorphism $E_R^{s,t} \subset E_r^{s,t}$ for $s < r \leq R < \infty$,

$$E_\infty^{s,t} = \bigcap_{s<r<\infty} E_r^{s,t};$$

5) there exist such subgroups $B^{s,t}$ of the group $\pi_{t-s}^S(X)$ that

$$\cdots \subset B^{s,t} \subset B^{s-1,t-1} \subset \cdots \subset B^{0,t-s} = \pi_{t-s}^S(X), \qquad E_\infty^{s,t} = B^{s,t}/B^{s+1,t+1};$$

6)
$$\bigcap_{t-s=m} B^{s,t} = K_m^S(X).$$

This theorem contains almost no information about the structure of the groups $B^{s,t}$. For the case $X = S^0$, Adams proved the assertion that these groups are arranged as follows. Let $\alpha \in \pi_{m+n}(S^n)$ $(m < n-1)$ define an element $\alpha' \in \pi_m^S(S^0)$. Then α' has filtration s, i.e. $\alpha' \in B^{s,t}$ and $\alpha' \notin B^{s+1,t+1}$, where $m = t - s$, if and only if in the Adams complex

$K_\alpha = e^{m+n+1} \underset{\alpha}{\cup} S^n$ there acts nontrivially a cohomology operation

$$Q : H^n(K_\alpha;\ Z_p) \to H^{n+m+1}(K_\alpha;\ Z_p)$$

of order s, and every cohomology operation of order $< s$ acting according to the same scheme is trivial in K_α.

We now give the general results that we shall need.

Lemma 1. *For any space X and prime p there exists a space X_p and a map $f^p : X \to X_p$ such that* $\operatorname{coker} f^p{*} = 0$, $\ker f^p{*} = K_*(\dot{X})$. *The space X_p, possessing the indicated properties, is determined uniquely up to singular homotopy type (we say that spaces have the same singular homotopy type if their natural systems are isomorphic).*

It follows from this lemma that the homotopy groups mod p of the spaces X and X_p are arranged in the same way.

Definition 1. A space X is called a **p-space** if it has the same singular homotopy type as X_p.

From the construction of the Adams sequence it follows:

Lemma 2. *The Adams sequences mod p, computed for the spaces X and X_p, are isomorphic, and this isomorphism is realized by the map $f^p{*}$.*

This lemma shows that, for the study of the Adams sequence, it is enough to regard X as a p-space.

Definition 2. A ***p*-system** of some p-space X is a system of fibrations

$$X_0 \overset{F_1}{\underset{\rho_1}{\longleftarrow}} X_1 \overset{F_2}{\underset{\rho_2}{\longleftarrow}} X_2 \overset{F_3}{\underset{\rho_3}{\longleftarrow}} \cdots, \tag{1}$$

satisfying the conditions:

1) the limiting space $\bar{X}$ of this system of fibrations has the same singular homotopy type as the space X;
2) the spaces $X_0, F_1, F_2, \ldots$ are direct products of spaces of type $K(Z_p, n)$ (the numbers n in one and the same product may be different);
3) the natural projections $p_j : \bar{X} \to X_j$ induce epimorphisms $(p_j)_*$ for all $j \geq 0$.

In the class of all p-systems of the space X one can introduce a relation of partial order. Namely, let $\alpha = \{X_i, \rho_i, F_i, p_i\}$ and $\beta = \{X_i', \rho_i', F_i', p_i'\}$ be two p-systems. Then $\alpha > \beta$ if, identifying the groups $\pi_*(X)$, $\pi_*(\bar{X})$, and $\pi_*(\bar{X}')$, $\ker(p_i)_* = \ker(p_i')_*$, $i < n$, and $\ker(p_n)_* \subset \ker(p_n')_*$. We can now formulate the following main theorem.

Theorem 2. *In the class of p-systems of a certain space X there is one and only one p-system (up to singular homotopy type) which is maximal with respect to the order introduced.*

Theorem 2 allows us to define the notion of a natural filtration that we need.

Definition 3. The **natural filtration** of a p-space is the sequence of subspaces of the space $\bar{X}$:

$$\bar{X} = Y_0 \supset Y_1 \supset \cdots \supset Y_s \supset \cdots,$$

where $Y_i = p_i^{-1}(F_i)$, $\{X_i, \rho_i, F_i, p_i\}$ is the maximal p-system of the space X, and $\bar{X}$ is its limiting space.

Definition 4. We shall call a filtration of the homotopy and homology groups of a p-space X **natural** if it is generated by the natural filtration of the space X.

We note that the inclusions $i_k : Y_k \subset \widetilde{X}$ induce monomorphisms $(i_k)_*$, i.e. the groups $\pi_*(Y_k)$ "realize" precisely all elements of filtration $\geq k$.

Digressing somewhat, we observe that the notion of a maximal p-system of the space X determines a method for computing the homotopy groups $\pi_i(X)$ by "killing cohomology operations" of various orders, in contrast with the classical method of "killing homotopy groups." This follows from the fact that the space X_s is a universal example for constructing certain operations of order $s+1$.

The following lemma makes it possible to determine the natural filtration in the groups $\pi_m^s(X)$.

Lemma 3. The elements $\alpha \in \pi_{m+n}(S^{nX})$ and $E\alpha \in \pi_{m+n+1}(S^{nX})$, where E is the suspension, have the same natural filtration if $m < n-1$.

Theorem 3. The filtration of the group $\pi_*^s(X)$ defined by the Adams sequence coincides with the natural filtration of this group.

We shall show how Theorem 3 can be proved on the basis of Theorem 2. Suppose that $\pi_i(X) = 0$ for $i < n$, where n is very large. In the contrary case one may consider the space S^{nX}. Let, further, $\{X_i, \rho_i, F_i, p_i\}$ be a maximal p-system of the space X and let the sequence

$$0 \leftarrow H^*(X; Z_p) \overset{\varepsilon}{\leftarrow} C_0 = \sum_{t\geq 0} C_{0,t} \overset{d}{\leftarrow} C_1 = \sum_{t\geq 0} C_{1,t} \overset{d}{\leftarrow} \cdots$$

be a free acyclic A-resolution of the A-module $H^*(X; Z_p)$. Since the A-module C_0 is free and, as we suppose, is of finite type, it can be realized up to dimension $2n-1$ as the cohomology module of some direct product B of Eilenberg–Mac Lane spaces. Therefore there exists a map, unique up to homotopy, $f : X \to B$, satisfying the condition $f^* = \varepsilon$ in dimensions $< 2n-1$, if one takes into account the identification of C_0 and $H^*(B; Z_p)$. We now replace the map f by the fibration $f : \widetilde{X} \to B$ and suppose that the fibre of this fibration is the space Φ. Then, as follows from the construction of the Adams sequence, the inclusion $\mu : \Phi \subset \widetilde{X}$ determines all elements of filtration 1 in the sense of Adams, i.e. these elements form the subgroup $\operatorname{im} \mu_*$.

Since $\operatorname{coker} \varepsilon = 0$, the space B can be constructed in the following way. Choose in the A-module $H^*(X; Z_p)$ a minimal homogeneous system of generators $u = \{u_i\}$, $i = 1, 2, \dots, N$, and with each element u_i associate the space $K_i = K(Z_p; \deg u_i)$. Putting

$$B = \sum_{i=1}^{N} K_i,$$

we obtain the required representation. The map $\varepsilon = f^*$ can then be specified by the relations $\varepsilon(\overline{v}_i) = u_i$, where $\overline{v}_i$ is the image in $H^*(B; Z_p)$ of the fundamental class v_i of the space K_i under the projection $B \to K_i$. We note at once that, if

$$X_0 = \sum_{i=1}^{l} K(Z_p; m_i),$$

then, according to the definition of a p-system, the images under p_0^* of the fundamental classes of the spaces $K(Z_p;m_i)$ (or their multiples modulo p) enter into the system u. Therefore, for $i \leq l$, we shall identify K_i and $K(Z_p;m_i)$.

We now realize the fibration $\tilde{f}$. For this purpose consider the fibration

$$\tau : \widetilde{X} \to \widetilde{X} \times \sum_{i>l} K_i = T,$$

which is the Whitney sum of fibrations whose Eilenberg–Mac Lane invariants are equal to $(u_i' - v_i')$, where u_i', v_i' are the images of the classes u_i, v_i, respectively, in $H^*(T;Z_p)$ under the corresponding

projections. The space $\overline{\overline{X}}$ has the same singular homotopy type as X. Indeed, $\overline{\overline{X}}$ is the Whitney sum of fibrations

$$\lambda_i : \overline{\overline{X}}_i \xrightarrow{E_\alpha} \overline{X},$$

whose fibers E_α are contractible, if one sets $\lambda_i = r_i \tau_i$, where the fibration

$$\tau_i : \overline{\overline{X}}_i \to \overline{X} \times K_i$$

is determined by the Eilenberg–MacLane invariant equal to

$$(u_i \otimes 1 - 1 \otimes v_i) \in H^*(\overline{X} \times K_i; Z_p),$$

and $r_i : \overline{X} \times K_i \to \overline{X}$ is the projection. The equivalence of the spaces $\overline{\overline{X}}$ and $\overline{X}$ is generated, obviously, by the mapping $r \circ \tau$, where $r : T \to \overline{X}$ is the projection. On the other hand, the fibration

$$p_0 \times 1 : T \to \overline{X} \times \left(\sum_{i>l} A_i \right) \to X_0 \times \left(\sum_{i>l} K_i \right) = B.$$

It turns out that

$$f' = (p_0 \times 1) \circ \tau : \overline{X} \to B$$

and there is a realization of the fibration $\tilde{f}$, since for $i > l$

$$f^*(v_i) = \tau^*(p_0 \times 1)^*(v_i) = \tau^*(v_i) = \tau^*(u_i) = (r \circ \tau)^*(u_i),$$

and, similarly,

$$f^*(v_i) = (r \circ \tau)^*(u_i)$$

for $i < l$. The fiber Φ' of the fibration f', as follows from the construction of f', can be represented in the form of the Whitney sum of fibrations

$$\nu_i : M_i \to \overline{Y}_1, \quad i > l,$$

where

$$\overline{Y}_1 = (r \circ \tau)^{-1}(Y_1)(Y_1 = p_1^{-1}(F_1)),$$

and ν_i is determined by the Eilenberg–MacLane invariant

$$i^*[(r \circ \tau)^*(u_i)],$$

where

$$\overline{Y}_i \to \overline{X}$$

is an inclusion. It follows from this that

$$\operatorname{im} \mu_* = \operatorname{im} i_*,$$

where $\mu' : \Phi' \subset \overline{X}$ is an inclusion, which shows that the elements of filtration $\geq l$ in the sense of Adams have natural filtration $\geq l$, and conversely.

Noting now that the elements of the group $\pi_*(\overline{Y}_1)$, whose natural filtration is ≥ 1, coincide under i_* with the elements of the group $\pi_*(X)$ whose natural filtration is ≥ 2, and similarly for the Adams filtration, we can continue our argument by induction. Theorem 3 is proved.

Adams' assertion. We shall now derive Adams' assertion from Theorem 3. Let

$$\alpha' \in \pi_m^S(S^0)$$

have filtration s in the sense defined above. Then one may suppose that α' is determined by a generating element α of some space of type

$$K(Z_p; n+m) \in F_i,$$

where $\{X_i, \rho_i, F_i, p_i\}$ is a maximal p-system of the sphere S^n. This means that α "kills" a certain element of

$$H^{m+1}(X_{i-1}),$$

i.e. an example of an operation of order $\leq i$. By virtue of the maximality of the p-system, the order of this operation P is exactly i. Attaching the element α leads to the operation P ceasing to be "killed," and hence, in the complex

$$e^{m+n+1} \cup_\alpha S^n$$

it will be nontrivial. Operations of order $< i$ will still remain trivial, i.e. $s = i$.

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Note: Figure translations are in progress. See original paper for figures.