MATHEMATICS

V. P. GURARII

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. P. GURARII

A GENERALIZATION OF THE FOURIER TRANSFORM AND OF THE WIENER–PALEY THEOREM

(Presented by Academician S. N. Bernstein on 21 X 1959)

§ 1. In the present paper we consider the question of extending the Fourier transform to the Hilbert space L^2_φ of functions, the scalar product in which is equal to

$$(f,\ g) = \int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{\varphi(x)}\, dx,$$

where the weight $\varphi(x)$ is a positive function on the real axis. The Fourier transform in such a space was first considered by N. I. Akhiezer (^{1}) under the assumption that $\varphi(z)$ is an entire function of zero order whose zeros lie in the strip $|\operatorname{Re} z| < A$. In that work the functions

$$\omega(z) = \prod\left(1 - \frac{z}{z_k}\right), \qquad \overline{\omega}(z) = \prod\left(1 - \frac{z}{\overline{z}_k}\right),$$

were constructed, where z_k are the zeros of $\varphi(z)$ lying in the half-plane $\operatorname{Im} z > 0$, and the following theorems were proved:

I. Every function $f(x) \in L^2_\varphi$ can be represented in the form

$$f(x) = \sum_{k=0}^{\infty} a_{kP} k(x) + \frac{\omega(x)}{\sqrt{2\pi}} \int_0^{\infty} h(t) e^{itx} dt + \frac{\overline{\omega}(x)}{\sqrt{2\pi}} \int_{-\infty}^{0} h(t) e^{itx} dt,$$

where $P_k(x)$ is an orthonormal sequence of polynomials in L^2_φ, $a_k = (f, P_k)$ $(k = 0, 1, ...)$, and $h(t) \in L^2(-\infty, \infty)$. Moreover, Parseval' s equality holds:

$$\|f\|^2 = \sum_{k=0}^{\infty} |a_k|^2 + \|h\|^2_{L^2}.$$

II. In order that the function $f(x) \in L^2_\varphi$ be an entire function of finite degree, less than or equal to σ, it is necessary and sufficient that $h(t)$ be equal to zero for $|t| > \sigma$.

We have proved analogous theorems, but under weaker restrictions on the function $\varphi(x)$. In addition, it is shown that any further weakening of the requirements on $\varphi(x)$ is impossible.

§ 2. We shall assume that $\varphi(z)$ is an entire function of zero order belonging to class $*A$. Let us note that under these restrictions on the weight, polynomials may fail to belong to L^2_φ.

---

* An entire function is called a function of class A if its zeros satisfy the inequality $\sum_k \left|\operatorname{Im} \frac{1}{a_k}\right| < \infty$.

In what follows we shall rely on the following theorem of N. I. Akhiezer (^{2}):

In order that an entire function $F(z)$ of finite degree σ be representable in the form $F(x) = |\Omega(x)|^2$, where $\Omega(z)$ is an entire function of finite degree $\sigma/2$ with roots in the half-plane $\operatorname{Im} z \geqslant 0$ $(\operatorname{Im} z \leqslant 0)$, it is necessary and sufficient that $F(z)$ be a function of class A, nonnegative on the real axis.

Applying this theorem to the weight $\varphi(x)$, we obtain that $\varphi(x) = |\omega(x)|^2$, where $\omega(z)$ is an entire function of zero degree with roots in the half-plane $\operatorname{Im} z > 0$. Let $\{z_k\}_1^\infty$ be the sequence of roots of $\omega(z)$. For simplicity of exposition we shall assume that the roots z_k are simple. Put $\overline{\omega}(z) = \overline{\omega(\overline{z})}$. The roots of $\overline{\omega}(z)$ lie in the half-plane $\operatorname{Im} z < 0$. Introduce the functions

$$\omega_k(z) = \sqrt{\frac{z_k - \overline{z}_k}{2\pi i} \frac{\omega(z)(z-\overline{z}_1)(z-\overline{z}_2)\cdots(z-\overline{z}_{k-1})}{(z-z_1)\cdots(z-z_{k-1})(z-z_k)}} \qquad (k = 1, 2, \dots);$$

these are entire functions of zero degree belonging to L^2_φ. It is easy to verify that $\{\omega_k(x)\}_1^\infty$ is an orthonormal sequence in L^2_φ.

Theorem 1. *In order that a function $f(x)$ belong to the space L^2_φ, it is necessary and sufficient that it be representable in the form*

$$f(x) = \sum_{k=1}^\infty a_k \omega_k(x) + \frac{\omega(x)}{\sqrt{2\pi}} \int_0^\infty h(t) e^{itx}\,dt + \frac{\overline{\omega}(x)}{\sqrt{2\pi}} \int_{-\infty}^0 h(t) e^{itx}\,dt, \qquad (1)$$

where $a_k = (f, \omega_k)$, $h(t) \in L^2(-\infty, \infty)$. In this case Parseval' s equality holds

$$\|f\|^2 = \sum_{k=1}^\infty |a_k|^2 + \|h\|^2_{L^2}. \qquad (2)$$

Proof. We shall confine ourselves to proving necessity, since sufficiency is obvious.

Consider three families of functions:

$$1.\ \omega_k(x)\ (k=1,2,\ldots). \qquad 2.\ \frac{\omega(x)}{x-z}\ (\operatorname{Im} z<0). \qquad 3.\ \frac{\overline{\omega}(x)}{x-w}\ (\operatorname{Im} w>0).$$

All these functions belong to the space L^2_φ, and the functions from the different systems are mutually orthogonal. Denote by H^0, H^+, H^-, respectively, the closures in L^2_φ of the linear spans of the families 1, 2, 3. These closures form mutually orthogonal subspaces in L^2_φ. Note that any function $f^+(x)$ belonging to H^+ has the form

$$f^+(x) = \frac{\omega(x)}{\sqrt{2\pi}} \int_0^\infty h(t) e^{itx}\, dt. \tag{3}$$

This becomes clear if one takes into account that the closure of the set of linear aggregates of the form $\sum_k \dfrac{c_k \omega_k(x)}{x-a_k}$ $(\operatorname{Im} a_k<0)$ in L^2_φ is equivalent to the closure in $L^2(-\infty,\infty)$ of the aggregates $\sum_k \dfrac{c_k}{x-a_k}$. Similarly, any function $f(x)\in H^-$ has the form

$$f^-(x) = \frac{\overline{\omega}(x)}{\sqrt{2\pi}} \int_{-\infty}^0 h(t) e^{itx}\, dt. \tag{3'}$$

We shall prove that the direct sum of the subspaces H^0, H^+, and H^- gives the whole space L^2_φ, i.e., that $H^0\oplus H^+\oplus H^- = L^2_\varphi$. To this end it is necessary to show that every function $f(x)\in L^2_\varphi$ orthogonal to each of the subspaces H^+, H^-, and H^0 is identically zero.

1. Let $f(x)\in L^2_\varphi$, and let $f(x)$ be orthogonal to any function from H^+; then, denoting by $\bar{L}^2$ (respectively, $\overset{+}{L}{}^2$) the space of functions from $L^2(-\infty,\infty)$ whose Fourier transform is equal to zero for $t>0$ (respectively $t<0$), one may assert that

$$\frac{f(x)}{\omega(x)} \in \bar{L}^2. \tag{4}$$

It is not difficult to prove the converse as well, namely: if (4) holds, then $f(x)$ is orthogonal to any function from H^+.

2. Analogously we obtain that the condition

$$\frac{f(x)}{\omega(x)} \in \overset{+}{L}{}^2$$

is necessary and sufficient for the function $f(x)$ to be orthogonal to any function from H^-.

Using the well-known Wiener-Paley theorem and results 1, 2, one can show that a function orthogonal to H^+ and H^- extends to the whole plane as an entire function of zero degree.

3. It remains to prove that an entire function $f(z)$ of zero degree, for which $f(x) \in L^2_\varphi$ and $(f, \omega_k) = 0$ $(k = 1, 2, \ldots)$, is identically zero. But from the equalities $(f, \omega_k) = 0$ $(k = 1, 2, \ldots)$, by Cauchy' s theorem it follows that $f(\bar{z}_k) = 0$. Therefore the function $\psi(z) = f(z)/\omega(z)$ is an entire function, and its degree is obviously equal to zero.

And since $\psi(x) \in L^2(-\infty, \infty)$, it follows that $\psi(z) \equiv 0$, and hence $f(z) \equiv 0$. Thus, we have proved that every function $f(x) \in L^2_\varphi$ is represented in the form

$$f(x) = f^0(x) + f^+(x) + f^-(x), \tag{5}$$

where

$$f^0(x) \in H^0, \quad f^+(x) \in H^+, \quad f^-(x) \in H^- \quad \text{and} \quad \|f\|^2 = \|f^0\|^2 + \|f^+\|^2 + \|f^-\|^2.$$

Moreover, we have proved that $\{\omega_k(x)\}_1^\infty$ is not only an orthonormal system, but also a basis in H^0. From (3), (3′), (5) the assertion of the theorem follows.

It should be noted that the series $\sum_k a_k \omega_k(z)$, under the condition $\sum_k |a_k|^2 < \infty$, converges uniformly in every finite domain of the complex plane.

§ 3. **Theorem 2.** *In order that $f(x) \in L^2_\varphi$ be an entire function of finite degree σ, it is necessary and sufficient that in the expansion (1) $h(t) = 0$ for $|t| > \sigma$.*

Proof. Here too we shall prove necessity, since sufficiency is obvious.

Preserving the notation for each of the parts of the expansion (5), it is easy to see that $f^+(x)$ is also an entire function of finite degree, not exceeding σ. From representation (3) it is clear that

$$\int_0^\infty h(t) e^{itx} \, dt$$

is an entire function whose degree also does not exceed σ. By the Wiener-Paley theorem $h(t) = 0$ for $t > \sigma$.

In exactly the same way we obtain that $h(t) = 0$ for $t < -\sigma$.

§ 4. **Theorem 3.** *Let the weight $\varphi(x)$ satisfy the following conditions:*

1) *There exists a function $\omega(z)$, analytic and having no zeros in the lower half-plane, such that $\varphi(x) = |\omega(x)|^2$.*
2) *Every function $f(x) \in L^2_\varphi$ has the representation*

$$f(x) = f_0(x) + \frac{\omega(x)}{\sqrt{2\pi}} \int_0^\infty h(t) e^{itx}\, dt + \frac{\overline{\omega}(x)}{\sqrt{2\pi}} \int_{-\infty}^0 h(t) e^{itx}\, dt,$$

where $f_0(x)$ is an entire function of zero degree, $f_0(x) \in L^2_\varphi$, $h(t) \in L^2(-\infty, \infty)$, $\overline{\omega}(z) = \omega(\overline{z})$; moreover $\|f\|^2 = \|f_0\|^2 + \|h\|^2_{L^2}$.

Then $\varphi(x)$ extends to the entire complex plane as an entire function of zero degree and of class A.

§ 5. Suppose that all polynomials belong to the space L^2_φ. In this case one may pose the question of the completeness of the system of polynomials in the space H^0 of entire functions of zero degree belonging to L^2_φ.

Theorem 4. *In order that the system of polynomials be complete in H^0, it is necessary and sufficient that, for all z and w, the equality*

$$\sum_{n=1}^\infty P_n(z)\overline{P_n(w)} = \frac{1}{2\pi i} \frac{\omega(z)\overline{\omega}(w) - \overline{\omega}(z)\omega(w)}{z - w},$$

hold, where $\{P_n(x)\}_1^\infty$ is an orthonormal sequence of polynomials in L^2_φ.

Received
20 X 1959

REFERENCES

[1] N. I. Akhiezer, DAN, 96, No. 5, 889 (1954). [2] N. I. Akhiezer, DAN, 63, No. 5, 475 (1948).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.