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Abstract

Full Text

AERODYNAMICS

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APPROXIMATE CONSTRUCTION OF ONE CLASS OF UNSTEADY TRANSONIC FLOWS

(Presented by Academician S. A. Khristianovich on 14 IV 1959)

Approximate equations describing steady transonic motions of a gas have been obtained and studied by many authors (¹⁻⁶). In (⁷) an equation was derived that is satisfied by the velocity potential of unsteady transonic flows when the variation of the flow parameters in time occurs sufficiently rapidly. In the present work an exact particular solution of this equation is obtained; it describes potential flows without shock waves, containing local supersonic zones varying with time. The shape of the nozzle is not prescribed in advance, but must be chosen in accordance with the constructed solution; generally speaking, it varies with time.

Spatial potential flows of an ideal gas satisfy the equations

$$\frac{\partial^2 \Phi}{\partial t^2} - a^2 \Delta \Phi + 2 \operatorname{grad} \Phi \operatorname{grad} \frac{\partial \Phi}{\partial t} + \frac{1}{2} \operatorname{grad} \Phi \operatorname{grad} (\operatorname{grad} \Phi)^2 = 0; \quad (1)$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\operatorname{grad} \Phi)^2 + \frac{a^2}{\varkappa - 1} = \text{const}, \quad (2)$$

where t is time; Δ is the Laplace operator; Φ is the velocity potential; a is the local speed of sound; \varkappa is the ratio of specific heats.

Consider the motion of a gas in the neighborhood of the critical section of a Laval nozzle, which has two planes of symmetry. We shall assume that the particle velocities in the flow differ only slightly in magnitude from the speed of sound, and that the angles between the direction of the velocity and the axis of the nozzle are small. Taking the x -axis of the cylindrical coordinate system x, r, ϑ to coincide with the axis of the nozzle and using the usual assumptions of the theory of transonic flows, from equations (1) and (2) we obtain (⁷)

$$-\frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \vartheta^2} - 2 \frac{\partial^2 \varphi}{\partial x \partial \tau} + \frac{1}{r} \frac{\partial \varphi}{\partial r} = 0. \quad (3)$$

In the equation obtained,

$$\frac{a_*}{x+1} \varphi(x, r, \vartheta, \tau) = \Phi(x, r, \vartheta, \tau) - a_* x \quad (\varphi \ll a_* x) \quad (4)$$

and the notation $\tau = a_* t$ has been used. The constant a_* in (4) is equal to the critical speed of sound, which it proves possible to introduce since the first term on the left-hand side of equation (2) may be neglected in the approximation under consideration.

Differentiating relation (3) with respect to x and introducing the new function

$$\frac{\partial \varphi}{\partial x} = u, \quad (5)$$

we obtain the equation

$$-\frac{1}{2} \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \vartheta^2} - 2 \frac{\partial^2 u}{\partial x \partial \tau} + \frac{1}{r} \frac{\partial u}{\partial r} = 0. \quad (6)$$

In papers (⁸⁻¹⁰) solutions of equation (6) were obtained which represent stationary mixed flows containing local supersonic zones in the immediate vicinity of the walls near the critical section of the nozzle. Using the results of (¹⁰), it is easy to see that the solution describing analogous unsteady gas motions may be sought in the form

$$u = -2c^{-1} \frac{d\lambda}{d\tau} + 4d^2 c^{-2} (1 + k^2 + 2k \cos 2\vartheta) r^2 + 4dc^{-2} g(\xi, \tau);$$

$$\xi = \lambda(\tau) + cx + d(1 + k \cos 2\vartheta) r^2. \quad (7)$$

Substituting expressions (7) into equation (6), it is easy to show that the function $\lambda(\tau)$ and the constants c , d , and k may be chosen arbitrarily, while the function $g(\xi, \tau)$ must satisfy the partial differential equation

$$\frac{\partial}{\partial \xi} \left(g \frac{\partial g}{\partial \xi} \right) + \frac{1}{2} d^{-1} c \frac{\partial^2 g}{\partial \xi \partial \tau} = \frac{\partial g}{\partial \xi} + a^2 \quad (a^2 = 1 + k^2). \quad (8)$$

Everywhere below we shall assume, for simplicity, that $c > 0$ and $d > 0$.

Let us now note that the components of the flow velocity in the coordinates v_x , v_r , and v_ϑ are given by the formulas

$$v_x = \frac{a_*}{x+1} \frac{\partial \varphi}{\partial x}; \quad v_r = \frac{a_*}{x+1} \frac{\partial \varphi}{\partial r}; \quad v_\vartheta = \frac{a_*}{x+1} \frac{1}{r} \frac{\partial \varphi}{\partial \vartheta}. \quad (9)$$

It follows from this that $v_x = a_* u/(x+1)$. Let us find the remaining two velocity components, v_r and v_ϑ . For this purpose we use the equations which express the condition that the flow be irrotational:

$$\frac{1}{r} \frac{\partial v_x}{\partial \vartheta} = \frac{\partial v_\vartheta}{\partial x}; \quad \frac{\partial v_x}{\partial r} = \frac{\partial v_r}{\partial x}; \quad \frac{\partial(rv_\vartheta)}{\partial r} = \frac{\partial v_r}{\partial \vartheta}. \quad (10)$$

From the first two equations of this system, taking account of formulas (7), we have

$$\begin{aligned} \frac{x+1}{a_*} v_\vartheta &= -16d^2 c^{-2} k x r \sin 2\vartheta - 8d^2 c^{-3} k g(\xi, \tau) r \sin 2\vartheta + \frac{1}{r} \chi_1(r, \vartheta, \tau); \\ \frac{x+1}{a_*} v_r &= 8d^2 c^{-2} (1+k^2 + 2k \cos 2\vartheta) x r + \\ &+ 8d^2 c^{-3} (1+k \cos 2\vartheta) g(\xi, \tau) r + \chi_2(r, \vartheta, \tau). \end{aligned} \quad (11)$$

Using the last of relations (10), we obtain an equation relating the functions $\chi_1(r, \vartheta, \tau)$ and $\chi_2(r, \vartheta, \tau)$ that are to be determined:

$$\frac{\partial \chi_1}{\partial r} = \frac{\partial \chi_2}{\partial \vartheta}. \quad (12)$$

The second equation which these functions satisfy is found by substituting expressions (11) into the equation of motion (3):

$$\begin{aligned} \frac{\partial \chi_2}{\partial r} + \frac{1}{r^2} \frac{\partial \chi_1}{\partial \vartheta} + \frac{1}{r} \chi_2 &= \\ &= 16d^2 c^{-3} \left[d(1+k^2)(1+k \cos 2\vartheta) r^2 + (1+k^2) \lambda - \frac{1}{4} d^{-2} c^2 \frac{d^2 \lambda}{d\tau^2} \right]. \end{aligned} \quad (13)$$

A particular solution of the system of equations (12) and (13) is determined by the formulas

$$\begin{aligned} \chi_1 &= r^4 \left[l \sin 4\vartheta - \frac{8}{3} d^3 c^{-3} k (1+k^2) \sin 2\vartheta \right]; \\ \chi_2 &= 8d^2 c^{-3} \left[(1+k^2) \lambda - \frac{1}{4} d^{-2} c^2 \frac{d^2 \lambda}{d\tau^2} \right] r + \\ &+ 4r^3 \left[-\frac{1}{4} l \cos 4\vartheta + \frac{4}{3} d^3 c^{-3} k (1+k^2) \cos 2\vartheta + d^3 c^{-3} (1+k^2) \right]. \end{aligned} \quad (14)$$

The remaining solutions of this system may, by using the principle of superposition, be expressed in terms of harmonic functions. To describe flows in nozzles whose cross sections have two axes of symmetry, one may use solution (14), since in this case the functions v_x and v_r will be even in ϑ , while the function v_ϑ will be odd.

The solutions obtained, which describe time-varying motions of the gas, pass into the solution of the steady problem ⁽¹⁰⁾ for $\lambda = 0$ and $g = g(\xi)$.

Let us now proceed to a more detailed investigation of the function $g(\xi, \tau)$. For this purpose, introducing the new independent variable $\eta = 2dc^{-1}\tau$, we integrate equation (8) with respect to ξ :

$$g \frac{\partial g}{\partial \xi} + \frac{\partial g}{\partial \eta} = g + a^2 \xi. \quad (15)$$

We write the system of ordinary differential equations corresponding to the quasilinear partial differential equation (15):

$$\frac{d\xi}{g} = \frac{dg}{g + a^2 \xi} = d\eta. \quad (16)$$

The first equation of this system is easily integrated ⁽¹⁰⁾:

$$\left(g + \frac{a^2 \xi}{q_1}\right)^{q_2} \left(g + \frac{a^2 \xi}{q_2}\right)^{-q_1} = e_1. \quad (17)$$

Here e_1 is an arbitrary constant, and the quantities q_1 and q_2 are expressed by the equalities

$$q_1 = \frac{1}{2} \left(1 - \sqrt{1 + 4a^2}\right); \quad q_2 = \frac{1}{2} \left(1 + \sqrt{1 + 4a^2}\right). \quad (18)$$

Using the solution obtained, one may find one more independent first integral of system (16). We write it in the form

$$\eta - \int g^{-1} d\xi = e_2, \quad (19)$$

where e_2 is an arbitrary constant, and the function g is expressed in terms of ξ by formula (17). The solution of the partial differential equation (15) can now be written in the form

$$\left(g + \frac{a^2 \xi}{q_1}\right)^{q_2} \left(g + \frac{a^2 \xi}{q_2}\right)^{-q_1} = e \left(\eta - \int g^{-1} d\xi\right). \quad (20)$$

The function e in relation (20) is arbitrary. For $e = \text{const}$, solution (20) passes into the solution describing the corresponding steady flows, which were investigated in detail in ⁽¹⁰⁾.

Potential flows with local supersonic zones are, generally speaking, impossible without shock waves ^(11–13). Even if for some nozzle such a flow proves possible, then with a small change in its shape a shock wave appears. Therefore expression (20) describes the development of local supersonic zones that arise as the pressure difference at the inlet and outlet of a nozzle increases, the walls of which

change with time. They must be chosen in accordance with the solution constructed. However, in works ^(8,9) it is shown that, in the stationary case, in passing from one regime, described by formulas (7), (11), (14), to another, the nozzle walls change only insignificantly.

In accordance with solution (20), the dimensions of the supersonic zone, which initially forms near the channel walls in the vicinity of the critical section, gradually increase. When the pressure difference at the inlet and in the exhaust part of the nozzle becomes sufficiently large, the local supersonic zone closes on the axis of the channel. In solution (20) the function e then tends to zero.

More general solutions than those indicated in the present work can also be obtained. For this purpose the quantities c , d , and k in formula (7) must be regarded as functions of time; the entire investigation is then not very greatly complicated. As was indicated above, for an approximate representation of various stationary regimes in a given nozzle one may use solution (7), with the constants entering into it remaining unchanged.

Since the theoretical results agree satisfactorily with the experimental data ⁽⁹⁾, one may expect that, for an approximate description of nonstationary processes in Laval nozzles as well, the solution constructed above can be applied, keeping the quantities c , d , and k constant in it.

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REFERENCES

1. F. I. Frankl, *Izv. AN SSSR, Ser. Mat.*, 9, No. 5, 387 (1945).
2. S. V. Falkovich, *Prikl. matem. i mekh.*, 10, issue 4, 503 (1946).
3. Th. von Karman, *J. Math. and Phys.*, 26, No. 3, 182 (1947).

4. M. J. Lighthill, *Proc. Roy. Soc. London, A*, 191, No. 1026, 323 (1947).
5. L. V. Ovsyannikov, *Vestn. LGU*, No. 6, 47 (1952).
6. K. G. Guderley, *Theorie schallnaher Strömungen*, Berlin—Göttingen—Heidelberg, 1957.
7. C. S. Lin, E. Reissner, H. S. Tsien, *J. Math. and Phys.*, 27, No. 3, 220 (1948).
8. S. Tomotika, K. Tamada, *Quart. Appl. Math.*, 7, No. 2, 381 (1950).
9. S. Tomotika, Z. Hasimoto, *J. Math. and Phys.*, 29, No. 2, 105 (1950).
10. O. S. Ryzhov, *Prikl. matem. i mekh.*, 23, issue 4 (1959).
11. A. A. Nikol'skii, G. I. Taganov, *Prikl. matem. i mekh.*, 10, issue 4, 481 (1946).
12. F. I. Frankl, *Prikl. matem. i mekh.*, 11, issue 1, 199 (1947).
13. G. Guderley, "On the Presence of Shocks in Mixed Subsonic—Supersonic Flow Patterns," *Adv. in Appl. Mech.*, 3, N. Y., 1953.

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