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Abstract

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ON THE ASYMPTOTICS OF SOME FUNCTIONALS IN A TWO-DIMENSIONAL RANDOM-WALK SCHEME

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1. Let a random walk be defined in the plane E, with probability density of the magnitude of a jump in one step $p_\varepsilon(x, y)$, depending on a small parameter ε ($\varepsilon > 0$). For a sufficiently smooth bounded function $f(x, y)$, given in the left half-plane $x \leqslant 0$, define the functional $u_\varepsilon(x, y) = u_\varepsilon(x, y; f)$ from the integral equation

$$P_\varepsilon[u_\varepsilon] \equiv \iint_E u_\varepsilon(x + \varepsilon\xi, y + \varepsilon\eta)\, p_\varepsilon(\xi, \eta)\, d\xi\, d\eta - u_\varepsilon(x, y) = 0 \quad (x > 0) \tag{1}$$

with the condition

$$u_\varepsilon(x, y; f) = f(x, y) \quad \text{for } x \leqslant 0. \tag{2}$$

The functional $u_\varepsilon(x, y; f)$ is the mathematical expectation of the value of the function f at the point of first exit into the half-plane $x \leqslant 0$ from the point (x, y) of the right half-plane.

Below an algorithm is presented for constructing the asymptotic expansion in powers of ε for the functional $u_\varepsilon(x, y; f)$ under the following assumptions:

1°. Equation (1) with condition (2) has a unique solution in the class of bounded functions*.

2°. There exist finite $N + 3$ moments of the distribution

$$\iint_E x p_\varepsilon(x, y)\, dx\, dy = \varepsilon a_{1,\varepsilon}; \qquad \iint_E y p_\varepsilon(x, y)\, dx\, dy = \varepsilon a_{2,\varepsilon};$$

$$\iint_E x^r y^s p_\varepsilon(x, y)\, dx\, dy = b_{rs,\varepsilon} \quad (2 \leqslant r + s \leqslant N + 3).$$

3°. The distribution $p_\varepsilon(x, y)$ is differentiable with respect to ε, so that the expansion

$$p_\varepsilon(x, y) = p_0(x, y) + \sum_{k=1}^{N} \varepsilon^k p_k(x, y) + \varepsilon^{N+1} p_{\varepsilon,N+1}(x, y), \tag{3}$$

holds, where $p_0(x, y)$ is a probability density with zero first moments.

From this condition there follow the asymptotic expansions for the moments:

$$a_{i,\varepsilon} = \sum_{k=0}^{N} \varepsilon^k a_{ik} + \varepsilon^{N+1} a_{i,N+1,\varepsilon} \quad (i = 1, 2);$$

$$b_{rs,\varepsilon} = \sum_{k=0}^{N} \varepsilon^k b_{rsk} + \varepsilon^{N+1} b_{rs,N+1,\varepsilon} \quad (2 \le r + s \le N + 3), \tag{4}$$

where $b_{110} b_{220} - b_{120}^2 > 0$.

2. **Theorem.** Under assumptions 1^0–3^0 there is an asymptotic expansion for the functional $u_\varepsilon(x, y; f)$:

$$u_\varepsilon(x, y; f) = \sum_{k=0}^{N} \varepsilon^k u_k(x, y) + \sum_{k=0}^{N-1} \varepsilon^{k+1} v_k\left(\frac{x}{\varepsilon}; y\right) + O(\varepsilon^{N+1}), \tag{5}$$

in which the functions $u_k(x, y)$ are determined successively from the elliptic differential equations

$$\mathbf{L}_0 u_0 \equiv \frac{1}{2} b_{110} \frac{\partial^2 u_0}{\partial x^2} + b_{120} \frac{\partial^2 u_0}{\partial x\, \partial y} + \frac{1}{2} b_{220} \frac{\partial^2 u_0}{\partial y^2} + a_{10} \frac{\partial u_0}{\partial x} + a_{20} \frac{\partial u_0}{\partial y} = 0, \tag{6}$$

$$\mathbf{L}_0 u_k = -\sum_{r=1}^{k} \mathbf{L}_r u_{k-r}, \tag{7}$$

where

$$\mathbf{L}_k = a_{1k} \frac{\partial}{\partial x} + a_{2k} \frac{\partial}{\partial y} + \frac{1}{2} b_{11k} \frac{\partial^2}{\partial x^2} + b_{12k} \frac{\partial^2}{\partial x\, \partial y} + \frac{1}{2} b_{22k} \frac{\partial^2}{\partial y^2} + \sum_{m=1}^{k} \sum_{\substack{r+s=m+2 \\ r,s>0}} \frac{b_{rs,k-m}}{r!\, s!} \frac{\partial^{m+2}}{\partial x^r \partial y^s} \tag{8}$$

with boundary conditions at $x = 0$:

$$u_0(0, y) = f(0, y); \qquad u_k(0, y) = -v_{k-1}(0; y) \quad (k = 1, 2, \dots, N); \tag{9}$$

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* Sufficient conditions for the fulfillment of 1° are given in the monograph of A. Ya. Khinchin (^{1}), Ch. 3, § 3.

the boundary-layer terms in x, $v_k(x/\varepsilon; y)$, are determined as solutions, vanishing at infinity, of integral equations on the half-line $z \geq 0$:

$$\mathbf{P}_0 v_0 \equiv \int_{-\infty}^{\infty} p_0(\xi) v_0(z+\xi; y)\, d\xi - v_0(z; y) = 0, \qquad p_0(\xi) = \int_{-\infty}^{\infty} p_0(\xi, \eta)\, d\eta; \tag{10}$$

$$\mathbf{P}_0 v_k = -\sum_{m=1}^{k} \mathbf{P}_m v_{k-m}, \tag{11}$$

where

$$\mathbf{P}_k = \sum_{r=0}^{k} \mathbf{P}_{k,r}, \qquad \mathbf{P}_{k,r} v(z; y) = \int_{-\infty}^{\infty} \frac{\partial^r}{\partial y^r} v(z+\xi; y) \int_{-\infty}^{\infty} \frac{\eta^r}{r!}\, p_{k-r}(\xi, \eta)\, d\xi\, d\eta, \tag{12}$$

with values for $z < 0$:

$$v_k(z; y) = v_k(0; y) + \frac{z^{k+1}}{(k+1)!} \frac{\partial^k f(0, y)}{\partial x^k} - \sum_{r=1}^{k+1} \frac{z^r}{r!} \frac{\partial^r u_{k+1-r}(0, y)}{\partial x^r}. \tag{13}$$

3. We shall give explanations of the algorithm formulated in the theorem. In the limiting passage as $\varepsilon \to 0$, from the solution of equation (1) with condition (2) to the solution of equation (6), degeneration of the boundary conditions occurs.

In accordance with the general idea of the asymptotic method in problems with degenerate boundary conditions (see (2)), the asymptotic expansion for the solution $u_\varepsilon(x, y)$ of equation (1) contains regular asymptotic terms $u_k(x, y)$ and boundary layers in x, $v_k(x/\varepsilon; y)$. Equations (6) and (7) for the regular terms of the asymptotics are obtained from the expansion of the integral operator $\mathbf{P}_\varepsilon$, which determines the original equation (1), in powers of ε:

$$\varepsilon^{-2} \mathbf{P}_\varepsilon = \mathbf{L}_0 + \varepsilon \mathbf{L}_1 + \cdots + \varepsilon^N \mathbf{L}_N + \varepsilon^{N+1} \mathbf{L}_{\varepsilon, N+1}. \tag{14}$$

We obtain this expansion by applying Taylor' s formula in both variables x and y and the expansion (4) to the expression $\mathbf{P}_\varepsilon[u(x, y)]$.

Applying Taylor's formula in one argument y and the expansion (3), we construct another expansion of the operator $\mathbf{P}_\varepsilon$ in powers of ε:

$$\mathbf{P}_\varepsilon = \sum_{k=0}^{N} \varepsilon^k \mathbf{P}_k + \varepsilon^{N+1} \mathbf{P}_{\varepsilon, N+1}, \tag{15}$$

which leads to equations (10) and (11) for the boundary layers. The values of the boundary layers on the negative half-axis $z < 0$ are determined in such a way that the discrepancy in satisfying condition (2) by the expansion (5) is of order $O(\varepsilon^{N+1})$. In this case the regular terms of the asymptotics $u_k(x, y)$ are continued into the left half-axis in a continuously differentiable way (see (3), the supplement). Expanding the difference

$$u_\varepsilon(\varepsilon z, y) - f(\varepsilon z, y) = u_0(\varepsilon z, y) - f(\varepsilon z, y) + \sum_{k=1}^{N} [u_k(\varepsilon z, y) - v_{k-1}(0; y)] +$$

$$+ \sum_{k=1}^{N-1} \varepsilon^{k+1} [v_k(z; y) - v_k(0; y)] + O(\varepsilon^{N+1})$$

by Taylor' s formula in z and combining terms with equal powers of ε, we obtain (13) (taking (9) into account).

4. From equations (10) and (11), the boundary layers can be determined by the factorization method (see (4)). Let

$$\rho_0(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} p_0(x)\, dx.$$

For the function $\rho_1(\lambda) = [1 - \rho_0(\lambda)]\lambda^{-2}$, factorization on the axis λ $(-\infty < \lambda < +\infty)$ is possible, i.e., there exists an expansion

$$\rho_1(\lambda) = \rho_+(\lambda)\rho_-(\lambda), \tag{16}$$

in which the functions $\rho_+(\lambda)$ and $\rho_-(\lambda)$ are holomorphic and nonzero inside the upper and lower half-planes, respectively. Expansion (16) makes it possible to reduce equation (10) for the homogeneous boundary layer to the form

$$\int_0^{\infty} p_-(\xi) v_0(z - \xi; y)\, d\xi = 0 \quad (z > 0), \tag{17}$$

where

$$\rho_-(\lambda) = \int_0^{\infty} e^{-i\lambda x} p_-(x)\, dx.$$

The inhomogeneous boundary layer $v_k(z; y)$, satisfying equation (11), is defined in the form of the sum

$$v_k(z;y) = v_{k0}(z;y) + \int_z^{\infty} (\xi - z) w_k(\xi;y)\, d\xi. \tag{18}$$

Here $w_k(z;y)$ is the solution of the inhomogeneous integral equation on the half-axis $z \geqslant 0$, equal to zero for $z < 0$:

$$\int_{-\infty}^{\infty} p_1(\xi - z) w_k(\xi;y)\, d\xi = \varphi_k(z;y) \qquad (z \geqslant 0), \tag{19}$$

where $\varphi_k(z;y)$ is the right-hand side of equation (11), and

$$\rho_1(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda z} p_1(z)\, dz;$$

$v^*_{k0}(z;y)$ is the homogeneous boundary layer, determined from equation (17), taking, for $z < 0$, the values

$$v^*_{k0}(z;y) = v_k(z;y) - \int_0^{\infty} (\xi - z) w_k(\xi;y)\, d\xi. \tag{20}$$

5. The justification of the algorithm presented is carried out by the method of upper and lower functions (see ([1]), Ch. III, § 3).

In conclusion, we note that, under an appropriate complication of the algorithm (see ([2,5])), a Markov random walk in the plane in a domain with a sufficiently smooth boundary can be considered.

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