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Abstract

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ON A NEW APPLICATION OF I. M. VINOGRADOV' S ESTIMATES TO THE THEORY OF THE RIEMANN ZETA-FUNCTION

(Presented by Academician I. M. Vinogradov on 3 VI 1960)

Let $N(\Delta, T)$ be the number of zeros of the function $\zeta(s)$, $s = \sigma + it$, lying in the rectangle $\Delta \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$. If Δ and T are regarded as varying independently of one another, $T > 1$ and $\Delta \in (1/2, 1)$, then all our information on the behavior of $N(\Delta, T)$ is expressed by inequalities of the form

$$N(\Delta, T) < c_1 \ln^{c_2} T \cdot T^{\varkappa(\Delta)(1-\Delta)}. \tag{1}$$

Here and below c_ν denote absolute positive constants, $\varkappa(\Delta) \geqslant 0$. Various methods for obtaining inequalities of type (1) give different estimates for the function $\varkappa(\Delta)$. It is known, for example, that

$$\varkappa(\Delta) \leqslant \min\left\{\frac{3}{2-\Delta},\ 2(1+2c)\right\}, \tag{2}$$

where

$$c = \lim_{t\to\infty} \frac{\ln|\zeta(1/2+it)|}{\ln t}$$

(see (1), Ch. IX, §§ 18–19). The best known value for c is 15/92 (see (1), Ch. V, § 16). Inequality (2) is used in the solution of certain problems connected with prime numbers (see (2,3)). If it were known that $\varkappa(\Delta) \leqslant 2$, then in the solution of certain number-theoretic problems one would obtain, qualitatively, the same results as follow from the assumption of the truth of the Riemann hypothesis on the zeros of the zeta-function. However, from (2) one obtains only that $\lim \varkappa(\Delta) = 2 + 0$ as $\Delta \to 1/2$. In the author' s paper (4) it was pointed out that the estimate of $\varkappa(\Delta)$ as $\Delta \to 1$ can be considerably improved by means of the method of trigonometric sums. Later P. Turán (5) obtained the estimate

$$\varkappa(\Delta) \leqslant 2 + 600(1-\Delta)^{0.01} \tag{3}$$

for Δ sufficiently close to 1 (in a note he gives, without proof, the better estimate $\varkappa(\Delta) \leqslant 2 + (1-\Delta)^{0.14}$).

In this note it is shown that a proper application of new results of I. M. Vinogradov (see (6)) leads to the estimate

$$\varkappa(\Delta) \leqslant 2 + c_3(1-\Delta)^{1/3}. \tag{4}$$

The proof of inequality (4) is based on the following lemmas. Below $\rho = \beta + i\gamma$ denotes a zero of the zeta-function lying in the rectangle $R\{\Delta \leqslant \sigma \leqslant 1,\ \frac{1}{2}T \leqslant t \leqslant T\}$.

Lemma 1.

$$\left|\sum_{n<x} n^{-\rho}\right| \leqslant c_4 x^{1-\beta} T^{-1} \quad \text{for } x \geqslant T.$$

Proof see (1), Chapter IV, § 3.

Lemma 2. For $k \geqslant 7$ and under the condition $(k+1)^{-1}\ln T \leqslant \ln x \leqslant k^{-1}\ln T$, we have

$$\left|\sum_{x<n\leqslant T} n^{-\rho}\right| \leqslant c_5 \ln T \cdot x^{1-c_6k^{-2}-\beta}.$$

Proof. This is a simple consequence of I. M. Vinogradov' s estimates of trigonometric sums (6) and of Korput' s estimates (see (1), Ch. V, Th. 11).

Let

$$f(s) = \sum_{y<n\leqslant z} a_n n^{-s}, \quad \text{where} \quad a_n = \sum_{d|n,\ d>y} \mu(d),$$

and $\mu(d)$ is the Möbius function.

Lemma 3. If for every zero $\rho \in R$ of the zeta-function the inequality $|f(\rho)| \geqslant 1/2$ holds, then the number $Q(\Delta, T)$ of zeros belonging to R does not exceed

$$c_7 \ln^8 T \cdot (Ty^{1-2\Delta} + z^{2-2\Delta}).$$

Proof follows trivially from Theorem 1 of paper (7).

On the basis of the lemmas stated above, the proof of the theorem reduces to the following. Put $y_1 = zT^{-1}$, $z > T > y > y_1$, and use the identity

$$\begin{aligned} 1 = &\sum_{n\leqslant y_1} \mu(n)n^{-\rho} \sum_{kn\leqslant z} k^{-\rho} + \sum_{y_1<n\leqslant y} \mu(n)n^{-\rho} \sum_{k\leqslant T} k^{-\rho} \\ &- \sum_{y_1<n\leqslant y} \mu(n)n^{-\rho} \sum_{zn^{-1}<k\leqslant T} k^{-\rho} + \sum_{y<n\leqslant z} \mu(n)n^{-\rho} \sum_{kn\leqslant z} k^{-\rho}. \end{aligned} \tag{5}$$

With the help of Lemma 1 we estimate

$$\left|\sum_{n\leqslant y_1}\mu(n)n^{-\rho}\sum_{kn\leqslant z}k^{-\rho}\right|+\left|\sum_{y_1<n\leqslant y}\mu(n)n^{-\rho}\sum_{k\leqslant T}k^{-\rho}\right|<\frac{1}{4} \tag{6}$$

under the conditions $z < T^2$, $\Delta > 3/4$, and $T > T_0$. It is enough to consider only

$$\Delta \leqslant 1 - c_9 \ln\ln T \cdot \ln^{-1} T.$$

Define k from the inequalities

$$\frac{c_6}{4}(k+1)^{-3} \leqslant 1-\Delta \leqslant \frac{c_6}{4}k^{-3}$$

(this is possible for values of Δ sufficiently close to 1), and put

$$y = zT^{-\frac{1}{k+1}}.$$

With the help of Lemma 2 we obtain

$$\left|\sum_{y_1<n\leqslant y}\mu(n)n^{-\rho}\sum_{zn^{-1}<k\leqslant T}k^{-\rho}\right|\leqslant c_7c_6^{-1}k^2\ln T\cdot z^{1-\Delta}T^{-c_6k^{-3}}<\frac{1}{4} \tag{7}$$

for sufficiently large T. From identity (5) and inequalities (6) and (7) we obtain $|f(\rho)| \geqslant 1/2$, and Lemma 3 can be applied. To estimate $\chi(\Delta)$, it is enough to consider the expression

$$Ty^{1-2\Delta} + z^{2-2\Delta}.$$

Its minimal value is obtained from the condition

$$Ty^{1-2\Delta} = z^{2-2\Delta}$$

or

$$Tz^{1-2\Delta}T^{\frac{2\Delta-1}{k+1}} = z^{2-2\Delta},$$

whence it follows that

$$z = T^{1+\frac{2\Delta-1}{k+1}},$$

and the estimate

$$\chi(\Delta) \leqslant 2\left(1+\frac{2\Delta-1}{k+1}\right) \leqslant 2+2\left[\frac{c_6}{4}(1-\Delta)\right]^{1/3}.$$

Putting $(2c_6)^{1/3} = c_3$, we obtain estimate (4).

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REFERENCES

1. E. K. Titchmarsh, *The Theory of the Riemann Zeta-Function*, IL, 1953.
2. N. G. Chudakov, K. A. Rodosskii, *Uspekhi Mat. Nauk*, 4, 21 (1949).
3. Yu. V. Linnik, *Izv. AN SSSR, ser. matem.*, No. 16, 503 (1952).
4. K. A. Rodosskii, *DAN*, 86, No. 6, 1069 (1952).
5. P. Turan, *Acta Math. Acad. Sci. Hungaricae*, 5, fasc. 3-4, 145 (1954).
6. I. M. Vinogradov, *Izv. AN SSSR, ser. matem.*, 22, 161 (1958).
7. K. A. Rodosskii, *Izv. AN SSSR, ser. matem.*, 19, 97 (1955).