MATHEMATICS

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Abstract

Full Text

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PROPER CORRELATION FUNCTIONS AND THEIR APPLICATIONS IN THE THEORY OF STATIONARY MARKOV PROCESSES

(Presented by Academician A. N. Kolmogorov on 13 II 1960)

1. Consider a stationary real process with continuous parameter $\{x_t;\ -\infty < t < \infty\}$ and suppose that the random variables x_t have means equal to zero and variances equal to one. Denote x_{t_1} by x, x_{t_1+t} by y; by stationarity, x and y have identical a priori distributions.

Definition 1. $r_1(t)$ is called the **maximal correlation function** of the process x_t, if for $t \neq 0$

$$|r_1(t)| = \sup_{f,g} |\mathbf{M}(f(x)g(y))| \tag{1}$$

in the class of functions f, g satisfying the conditions

$$\mathbf{M}f(x) = \mathbf{M}g(y) = 0; \qquad \mathbf{M}f^2(x) = \mathbf{M}g^2(y) = 1, \tag{2}$$

where $\mathbf{M}$ denotes mathematical expectation. (For $t = 0$, $r_1(t)$ is completed by the condition $r_1(0) = 1$.)

Remark. If the functions f_1 and g_1 at which the upper bound (1) is attained are linear, then $r_1(t)$ coincides with the ordinary correlation function $r(t)$, where

$$r(t) = \mathbf{M}(xy), \qquad t \neq 0. \tag{3}$$

The properties of $r_1(t)$ are analogous to the properties of the maximal correlation coefficient ([1,2]).

2. We now considerably narrow the class of processes under consideration, assuming that the law of the joint distribution of x and y has density $p(t, x, y)$, that this density is symmetric,

$$p(t, x, y) = p(t, y, x) \tag{4}$$

in the region $\Omega = [a \leq x; \;\; y \leq b]$ and for all $t \neq 0$ satisfies the boundedness condition

$$\iint_{(\Omega)} \frac{p^2(t, x, y)}{p(x)p(y)} \, dx \, dy < \infty, \tag{5}$$

where

$$p(x) = \int_a^b p(t, x, y) \, dy, \qquad p(y) = \int_a^b p(t, x, y) \, dx. \tag{6}$$

By virtue of (4) and (5), the upper bound (1) is attained on identical functions, i.e. $g_1(y) = f_1(y)$.

Definition 2. The sequence of eigenvalues $r_k(t)$, $k = 1, 2, ...$, of the kernel

$$\frac{p(t, x, y)}{\sqrt{p(x)p(y)}} \tag{7}$$

is called the sequence of intrinsic correlation functions of the process x_{t_1}, $t \neq 0$.

Definition 3. If the eigenfunctions of the kernel (7) $\{\varphi_k(x), \varphi_k(y)\}$ do not depend on t, then the process is called maximally stationary.

Definition 4. The process x_{t_1} is called C-continuous if all $r_k(t)$ are continuous; in particular,

$$\lim_{t \to 0} r_k(t) = 1, \qquad k = 1, 2, ... \tag{8}$$

3. Suppose that $p(t, x, y)$, in addition to the symmetry condition (4) and the constraint (5), satisfies the Markov equation

$$p(t_1 + t_2, x, y) = \int_a^b \frac{p(t_1, x, z) \, p(t_2, z, y)}{p(z)} \, dz; \tag{9}$$

then this density completely determines a stationary Markov process.

A consequence of the definitions introduced and equation (9) is

Theorem 1. In order that a symmetric two-dimensional density $p(t, x, y)$, satisfying the constraint (5), be able to define a continuous Markov process, it is necessary and sufficient that the following conditions be fulfilled:

a) the eigenfunctions of the kernel (7) do not depend on t, i.e. the process is maximally stationary;

b) the intrinsic correlation functions have the form (for $k = 1, 2, \ldots$)

$$r_k(t) = e^{-\lambda_k t}, \qquad t \geq 0, \qquad 0 < \lambda_1 \leq \lambda_2 \leq \cdots. \tag{10}$$

4. Suppose now that the limits exist

$$\begin{aligned} A(x) &= \lim_{t \to 0} \frac{1}{t} \int_a^b (y - x) \frac{p(t, x, y)}{p(x)}\, dy, \\ B(x) &= \lim_{t \to 0} \frac{1}{t} \int_a^b (y - x)^2 \frac{p(t, x, y)}{p(x)}\, dy \end{aligned} \tag{11}$$

and that the transition probability density

$$f(t, x, y) = \frac{p(t, x, y)}{p(x)} \tag{12}$$

satisfies the well-known equations of A. N. Kolmogorov ([3])

$$\frac{\partial f}{\partial t} = A(x) \frac{\partial f}{\partial x} + \frac{1}{2} B(x) \frac{\partial^2 f}{\partial x^2}, \tag{I}$$

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}[A(y) f] + \frac{1}{2} \frac{\partial^2}{\partial y^2}[B(y) f]. \tag{II}$$

On the other hand, according to Theorem 1, for $f(t, x, y)$ one obtains the bilinear expansion

$$f(t, x, y) = p(y) \left[1 + \sum_{k=1}^{\infty} \varphi_k(x) \varphi_k(y) e^{-\lambda_k t} \right]. \tag{13}$$

Substituting (13) into (I) and (II) and using the orthogonality of $\varphi_k(x)$ with weight $p(x)$, we obtain the ordinary differential equations

$$\frac{1}{2} B(x) \varphi_k''(x) + A(x) \varphi_k'(x) + \lambda_k \varphi_k(x) = 0, \qquad k = 1, 2, \ldots; \tag{14}$$

$$B(x) p'(x) + [B'(x) - 2A(x)] p(x) = 0. \tag{15}$$

From (15) we find an explicit expression for $p(x)$:

$$p(x) = \frac{\gamma}{B(x)} \exp\left[\int \frac{2A(x)}{B(x)}\, dx\right], \tag{16}$$

where γ is a normalizing constant. (In particular, the process can be Gaussian only if $A(x)$ is linear and $B(x) = \text{const.}$)

5. Suppose, for example, that $\varphi_k(x)$ is a polynomial of degree k, $k = 1, 2, ...$, and that the domain Ω is infinite. In this case, using (13) and the orthogonality of the eigenfunctions, we immediately obtain

$$\begin{aligned} A(x) &= -\lambda_1 x, \\ B(x) &= (2\lambda_1 - \lambda_2)x^2 + c(\lambda_2 - \lambda_1)x + \lambda_2, \end{aligned} \tag{17}$$

where

$$c = Mx^3 \tag{18}$$

is the coefficient of asymmetry of the a priori distribution of x_{t_1} (since $Mx_{t_1} = 0$, and $Mx_{t_1}^2 = 1$).

Since $B(x) \geq 0$ on an infinite domain, the condition

$$\lambda_2 \leq 2\lambda_1 \tag{19}$$

is necessary.

On the other hand, substituting into (14), instead of $\varphi_k(x)$, a polynomial of degree k with undetermined coefficients, and instead of $A(x)$ and $B(x)$ their expressions (17), we find

$$\lambda_k = k\left[\frac{k-1}{2}\lambda_2 - (k-2)\lambda_1\right]. \tag{20}$$

Expression (20), when $\lambda_2 < 2\lambda_1$, becomes negative for sufficiently large k; consequently, in inequality (19) only the equality sign is possible, and then in general

$$\lambda_k = k\lambda_1, \qquad k = 1, 2, \dots, \tag{21}$$

and equation (14) is reduced to the form

$$\left(\frac{c}{2}x + 1\right)\varphi_k''(x) - x\varphi_k'(x) + k\varphi_k(x) = 0, \qquad k = 1, 2, \dots, \tag{22}$$

where

$$p(x) = \frac{\gamma_1}{cx+2} \exp\left[-\int \frac{2x}{cx+2}\,dx\right], \qquad cx+2 \geq 0, \quad c \geq 0, \quad \gamma_1 = \text{const.} \tag{23}$$

6. Let $c = 0$; then $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and $\varphi_k(x)$ are Hermite polynomials, and the process is Gaussian, while for $t > 0$

$$p(t,x,y) = \frac{1}{2\pi\sqrt{1-e^{-2\lambda_1 t}}} \exp\left[-\frac{x^2+y^2-2e^{-\lambda_1 t}xy}{2(1-e^{-2\lambda_1 t})}\right]. \tag{24}$$

is the density of a normal correlation with positive correlation coefficient $R = e^{-\lambda_1 t}$.

7. For $c > 0$, equation (22) defines generalized Laguerre polynomials, orthogonal on the interval $-\frac{2}{c} \leqslant x < \infty$ with weight

$$p(x) = \gamma_2 e^{-\frac{2}{c}x} \left(x + \frac{2}{c}\right)^{\frac{4}{c^2}-1}, \qquad \gamma_2 = \text{const.} \tag{25}$$

The corresponding density of the joint probability distribution has been studied in detail in paper [4]; we note here only that the characteristic function of this distribution has the form

$$\varphi_c(\tau_1,\tau_2) = \frac{\exp\left[-\frac{2i}{c}(\tau_1+\tau_2)\right]}{\left[1-\frac{c}{2}i(\tau_1+\tau_2)-\frac{c^2}{4}\left(1-e^{-\lambda_1 t}\right)\tau_1\tau_2\right]^{\frac{4}{c^2}}}, \tag{26}$$

and as $c \to 0$ it converges to the characteristic function of the normal distribution (24).

Remark. If $c < 0$, then the corresponding correlation dependence between x and y is specified in the quadrant $-\infty < x;\ y \leqslant \frac{2}{c}$, and a simultaneous change of signs of x and y reduces this case to the one already considered.

8. In items 5–7 the following assertion has been proved.

Theorem 2. *The normal correlation (24) and the correlation with characteristic function (26), constructed from generalized Laguerre polynomials, exhaust the class of densities determining a continuous stationary Markov process and such that the eigenfunctions of the kernel (7) constitute a complete system of polynomials orthogonal on an infinite interval.*

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Note: Figure translations are in progress. See original paper for figures.