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Abstract

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ON THE STABILITY OF PERIODIC MOTIONS

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The paper contains two results: a generalization of a theorem of A. A. Andronov and A. A. Witt is given, and the case of a canonical system of a special form is considered.

1. The generalization of the theorem of A. A. Andronov and A. A. Witt (1) on the stability of periodic motions of autonomous systems has been studied by a number of authors (2,3). The present work gives a solution of the indicated question that is more general than in (3): the case is considered in which the period of the solution depends on arbitrary constants.

Statement of the problem. Consider an autonomous system of n equations

$$\frac{dx_s}{dt} = X_s(x_0, x_1, \dots, x_{n-1}), \qquad s = 0, 1, \dots, n-1. \tag{1}$$

The functions $X_s(x_0, x_1, \dots, x_{n-1})$ are assumed to be continuous in some domain G of the space of the variables $x_0, x_1, \dots, x_{n-1}$ and to have, with respect to all variables $x_0, x_1, \dots, x_{n-1}$, continuous partial derivatives up to and including the second order.

Let system (1) admit in the domain G a family of periodic solutions depending on $k+1$ parameters $h_0, h_1, \dots, h_k$, belonging to some domain.

Owing to the autonomy of system (1), one of the constants, for example h_0, may be introduced as a phase shift, writing the solution in the form

$$x_s = \varphi_s(\omega t + h_0, h_1, \dots, h_k), \tag{2}$$

$$-\infty \le h_0 \le +\infty, \tag{3a}$$

$$|h_i| < H, \qquad H = \text{const}, \quad i = 1, 2, \dots, k, \tag{3b}$$

where the φ_s are periodic in ωt with period 2π, and, consequently, the solution has in t the period

$$T = T(h_1, \dots, h_k) = \frac{2\pi}{\omega(h_1, \dots, h_k)}, \qquad \omega(h_1, \dots, h_k)t = \psi(t, h_1, \dots, h_k). \tag{4}$$

What is new here in comparison with (^{3}) is the assumption that the period of the solution depends on the parameters $h_1, \dots, h_k$, which belong to the domain (3b).

We investigate the stability of the solutions of the family (2). From the indicated family (2) choose the solution corresponding to the parameter values

$$h_0 = 0, \qquad h_1 = 0, \dots, h_k = 0,$$

and take it as the unperturbed motion.

We shall set up the equations of the perturbed motion, assuming that

$$x_s = \varphi_s(\omega(0, 0, \dots, 0)t, 0, \dots, 0) + \xi_s,$$

$$s = 0, 1, \dots, n-1, \quad \omega(0, 0, \dots, 0) = \omega_0, \tag{5}$$

and consider their first approximation

$$\frac{d\xi_s}{dt} = p_{s0}\xi_0 + p_{s1}\xi_1 + \dots + p_{sn-1}\xi_{n-1}, \qquad p_{ij}(t) = \left(\frac{\partial X_i}{\partial x_j}\right)_0. \tag{6}$$

The $p_{ij}(t)$ depend on t through the solution $\varphi(\omega_0 t, 0, \dots, 0)$, i.e., they are computed for the unperturbed motion and therefore are periodic functions of period T_0.

Since equations (1) admit the solution (2), equations (6) admit a solution of the form

$$\xi_s = \left\{\frac{\partial \varphi_s}{\partial h_0}\right\}_0 h_0 + \left\{\frac{\partial \varphi_s}{\partial h_1} + \frac{\partial \varphi_s}{\partial \psi}\omega'_{h_1} t\right\}_0 h_1 + \dots + \left\{\frac{\partial \varphi_s}{\partial h_k} + \frac{\partial \varphi_s}{\partial \psi}\omega'_{h_k} t\right\}_0 h_k, \tag{7}$$

depending on $k+1$ arbitrary constants taken from the domain (3a), (3b), i.e., the variational equations (6) have $k+1$ independent particular solutions

$$\xi_{s0} = \left\{\frac{\partial \varphi_s}{\partial h_0}\right\}_0; \qquad \xi_{sj} = \left\{\frac{\partial \varphi_s}{\partial h_j} + \frac{\partial \varphi_s}{\partial \psi}\omega'_{h_j} t\right\}_0,$$

$$s = 0, 1, \dots, n-1; \qquad j = 1, 2, \dots, k, \tag{8}$$

where the subscript 0 means that the quantity is evaluated at the point

$$h_0 = 0, \qquad h_1 = 0, \dots, h_k = 0. \tag{9}$$

The solution ξ_{s0} is periodic in t, and this, according to Poincaré, implies that at least one root of the characteristic equation has modulus equal to unity. The remaining solutions $\xi_{sj}(t)$, $j = 1, 2, \dots, k$, are, generally speaking, nonperiodic. However, despite their nonperiodicity, it can be shown that the following holds:

Lemma. *If the system of variational equations (6) has $k+1$ particular solutions of the form (8), then the corresponding characteristic equation has a root with modulus equal to unity of at least $(k+1)$-fold multiplicity.*

In the proof of the lemma we use the definition of the characteristic equation given by A. M. Lyapunov, which is obviously applicable in this case, since the coefficients $p_{ij}(t)$ are periodic functions of t with period T_0. The $k+1$ solutions of the fundamental system are known to us; therefore $n(k+1)$ elements of the characteristic determinant can be written out explicitly, which makes it possible to verify the validity of the lemma directly. The method of proof was proposed by V. M. Volosov.

We assume the remaining roots to have moduli less than unity. We shall suppose that

$$T = T(h_1, \dots, h_k) = T_0(1+\alpha_h), \qquad \alpha_h = C_1^{(1)} h_1 + C_2^{(1)} h_2 + \dots + C_k^{(1)} h_k + \beta_h, \tag{10}$$

where $C_i^{(1)}$ are certain unknown but quite definite constants, and β_h has, with respect to the h_i, order of smallness not lower than the second. This form of T is a consequence of the requirements imposed on the right-hand sides of the equations of the original system.

In equations (1) let us perform the change of variable

$$t = \tau \frac{T}{T_0} = \tau(1 + \alpha_h). \tag{11}$$

Then equations (1) will take the form

$$\frac{dx_s}{d\tau} = X_s(x_0, x_1, \dots, x_{n-1})(1 + \alpha_h), \qquad s = 0, 1, \dots, n-1. \tag{12}$$

To these equations there corresponds a family of periodic solutions

$$x_s = \varphi_s(\omega_0\tau + h_0,\ h_1, \dots, h_k), \tag{13}$$

having, with respect to τ, period $T_0 = 2\pi/\omega_0$.

The problem of the stability of (2) is reduced to the problem of the stability of (13); moreover, instead of equations (6) one must consider the equations

$$\frac{d\xi_s}{d\tau} = \{p_{s0}\xi_0 + p_{s1}\xi_1 + \dots + p_{sn-1}\xi_{n-1}\}(1 + \alpha_h), \tag{14}$$

which satisfy all the conditions imposed on equations (6).

Thus, instead of solutions (2) of period T_h, satisfying the system of equations (1), one may investigate for stability the family of periodic solutions (13) of period T_0, satisfying the system of equations (12).

Using further the results of $^{(3)}$, we arrive at the conclusion that the following is true:

Theorem 1. *If, under all the conditions listed above, the equations of the first approximation of the differential equations of perturbed motion have $n - k - 1$ characteristic exponents with negative real parts, then the chosen unperturbed motion from the family of periodic solutions (2) is stable, and every perturbed motion not belonging to the family (2), sufficiently close at the initial instant of time to the unperturbed one, as t increases without bound approaches without bound one of the periodic motions.*

If, however, among the $n - k - 1$ characteristic exponents there is at least one with positive real part, then the indicated motions are unstable.

Thus, the theorem in its first part asserts that stability holds on the manifold of periodic solutions, while outside the manifold there is asymptotic stability with respect to a solution of the indicated manifold.

2. We shall now investigate the case of a nonlinear canonical system of a special form.

Statement of the problem. Let the canonical system of $2k$ equations

$$\frac{dx_s}{dt} = \frac{\partial H}{\partial y_s}, \qquad \frac{dy_s}{dt} = -\frac{\partial H}{\partial x_s}, \qquad s = 1,\ 2, \dots,\ k, \tag{15}$$

have a family of periodic solutions of period T_0

$$x_s = \varphi_s(t, h_1, \dots, h_k), \qquad y_s = \psi_s(t, h_1, \dots, h_k), \tag{16}$$

depending on k parameters belonging to the domain (36).

$H(t, x_1, \dots, x_k, y_1, \dots, y_k)$ is analytic in all the variables $x_1, \dots, x_k, y_1, \dots, y_k$ in the domain of the periodic solution under consideration. The coefficients of its expansion in these variables are periodic functions of t of period T_0.

Let the system (15) have $k-1$ analytic integrals

$$H_1 = 0, \qquad H_2 = 0, \dots, \ H_{k-1} = 0, \tag{17}$$

where H_s is a homogeneous form of degree s in the variables $x_1, \dots, x_k, y_1, \dots, y_k$ with periodic coefficients, i.e. the expansion of H has the form

$$H = H_k + H_{k+1} + \cdots \tag{18}$$

The solutions of the family (16) are investigated for stability.

We form the equations of perturbed motion under the assumption that

$$x_s = \varphi_s(t, 0, \dots, 0) + \xi_s, \qquad y_s = \psi_s(t, 0, \dots, 0) + \eta_s,$$

and consider their first approximation. The special form of the function H leads to the fact that k roots of the characteristic determinant, correspondi-

of the equation in variations satisfy the equality $|\rho| = 1$; moreover, the solution (16) ensures the fulfillment of the indicated equality also for k roots of the characteristic determinant. Suppose that the characteristic roots in these two groups do not coincide, i.e., the equality $|\rho| = 1$ holds for all $2k$ roots of the characteristic equation.

Assuming that all multiple roots are such that the number of groups of solutions corresponding to each of them is equal to its multiplicity, let us apply to the equations of the perturbed motion the transformation of N. G. Chetayev [4], which in the present case is a certain linear transformation with bounded coefficients. Then the system of equations of the perturbed motion will take the form

$$\frac{dz_m}{dt} = Z_m(t, z_1, \dots, z_k), \qquad m = 1, 2, \dots, k, \tag{19}$$

where $Z_m(t, z_1, \dots, z_k)$ are analytic functions of the variables $z_1, \dots, z_k$ with periodic coefficients, containing in their expansion with respect to the indicated variables no terms below second order.

From the form of the solution (16) and the transformation applied, we conclude that the solution of system (19) can be represented in the form

$$z_m = z_{m1}^{(1)} h_1 + z_{m2}^{(1)} h_2 + \cdots + z_{mk}^{(1)} h_k + z_{m1}^{(2)} h_1^2 + \cdots \tag{20}$$

At least one series of the indicated form must exist, as a consequence of the assumption of the existence for the original system of the solution (16).

Let us require that the condition

$$z_m(0) = \sum_{i=1}^{k} h_i = c \tag{21}$$

be satisfied.

Since the right-hand sides of equations (19) are analytic with respect to $z_1, \dots, z_k$, the solution of these equations determined by the initial condition (21) can be expanded in a series in c, convergent for

$$|c| = \left|\sum_{i=1}^{k} h_i\right| \leqslant \sum_{i=1}^{k} |h_i| \leqslant kH = \alpha > 0.$$

The number α is chosen so that the indicated series converges for all values of t on the interval $[0, T_0]$, where T_0 is the period of the solution.

It can be shown that the series (20) is identical to the series realizing the expansion of the solution $z_m(t, h_1, \dots, h_k)$ with respect to the initial value c. The terms of this series are periodic functions of t of period T_0, owing to the periodicity of the solution (16) of the original system. Thus, this solution is stable, but not asymptotically; that is, the following theorem is valid:

Theorem 2. *Under all the conditions listed above, the solutions of the system* $\varphi_s(t, 0, \dots, 0)$, $\psi_s(t, 0, \dots, 0)$ *and* $\varphi_s(t, h_1, \dots, h_k)$, $\psi_s(t, h_1, \dots, h_k)$ *are stable for sufficiently small* $h_1, \dots, h_k$. *If, however, at least one root of the characteristic equation has modulus different from unity, then the indicated solutions are unstable.*

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