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Abstract

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ON THE COEFFICIENTS OF CONVERGENT ORTHOGONAL SERIES WITH RESPECT TO COMPLETE SYSTEMS

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In a paper of Orlicz ([1]) (see also ([2]), p. 174) the following theorem is proved:

Let $\{\varphi_n(x)\}$ *be a complete orthonormal system of functions. Then almost everywhere on* $[0,1]$

$$\sum_{n=1}^{\infty} \varphi_n^2(x) = \infty. \quad (1)$$

It remained unknown, however, how definitive this assertion is, i.e., whether the exponent in (1) can be increased. Moreover, the following question remained unresolved:

Does there exist a complete orthonormal system $\{\varphi_n(x)\}$ **such that** $\varphi_n(x) \to 0$ $(n \to \infty)$ **almost everywhere or at least on a set of positive measure?**

It is clear that a negative answer to this question would imply that the terms of the series in (1) certainly do not tend to zero, and Orlicz' s theorem would admit strengthening.

Connected with this same question is the problem of the coefficients of convergent orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \quad (2)$$

with respect to complete systems. Indeed, as P. L. Ulyanov ([3]) showed, if the series (2) with respect to a complete system converges unconditionally almost everywhere on a set of positive measure, then

$$\lim_{n\to\infty} |c_n| = 0. \quad (3)$$

Naturally the question arose: is condition (3) also necessary for the ordinary convergence almost everywhere of a series with respect to a complete system? In particular, can the coefficients of a convergent orthogonal series with respect to a complete system tend to infinity? We note that if there existed no complete systems tending to zero, then the necessity of condition (3) would be obvious. However, as will be shown below, condition (3) is not necessary for the convergence of a series with respect to a complete system. Therefore there naturally arises the question of finding necessary and sufficient conditions in order that a given sequence c_n be the sequence of coefficients of a convergent orthogonal series with respect to a complete system.

An analogous question can be formulated for Fourier series. The solution of all the questions indicated above follows from the results of § 3. This section contains a necessary and sufficient condition (in the class of complete systems) on the coefficients of a convergent orthogonal series. The method of proof is based on the construction of the lemma of § 1. This method turns out to be useful in the study of some other questions connected with com–

systems. Thus, in particular, the question of Fourier coefficients with respect to complete systems (§ 4), and some others, can be resolved.

§ 1. **The main lemma.** We formulate a lemma that plays the principal role in obtaining the subsequent results.

Lemma. Let the interval $[0,1]$ be partitioned into a sum of nonintersecting sets E_j of positive measure $(j = 1, 2, ...)$. Suppose that on each of them an orthonormal system consisting of bounded functions $\{\psi_k^{(j)}(x)\}$ $(k = 1, 2, \dots;\ |\psi_k^{(j)}(x)| < M_{kj})$ is given, complete in $L^p(E_j)$ $(1 \le p < \infty)$. Let a sequence of orthogonal matrices $A_n = \|a_{rj}^{(n)}\|$ of order l_n be given $(n = 1, 2, \dots;\ l_{n+1} > l_n)$. Denote $\alpha_n = \sum_{k=1}^{n} l_k$. Put

$$\varphi_i(x) = \begin{cases} a^{(n_i)}_{i-\alpha_{n_i-1},\,j}\psi^{(j)}_{i-m_j}(x), & 1 \le j \le l_{n_i}, \\ 0, & j > l_{n_i}, \end{cases}$$

where n_i and m_j are determined from the inequalities

$$\alpha_{n_i-1} < i \le \alpha_{n_i}, \qquad l_{m_j} < j < l_{m_j+1}$$

(where we set $\alpha_0 = l_0 = 0$), $x \in E_j$ $(j = 1, 2, ...)$.

Then the functions $\{\varphi_i(x)\}$ form an orthonormal system complete in $L^p[0,1]$.

Remark 1. As we see, the lemma gives a construction of complete systems on $[0,1]$, starting from complete systems on E_j, where

$$\bigcup_{j=1}^{\infty} E_j = [0,1]$$

and $E_i \cap E_j = 0$ for $i \neq j$. In this connection we note that another scheme (and only for the case of completeness in L^2) was proposed by Kostyuchin ((2), p. 86), but, as is easily verified, it is unsuitable for our purposes—the construction of complete systems tending to zero.

§ 2. **An example of a complete orthonormal system tending to zero.** We define inductively a sequence of matrices

$$B_n = \|b_{r,k}^{(n)}\|$$

as follows:

$$B_1 = (1), \qquad B_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Suppose B_n has been defined. Then set

$$B_{n+1} = \begin{pmatrix} B_n & B_n \\ -B_n & B_n \end{pmatrix},$$

where by $-B_n$ we mean the matrix whose elements are opposite to the elements of the matrix B_n. To each natural number i assign the number p_i, determined from the inequalities

$$2^{p_i - 1} \leq i < 2^{p_i}.$$

Let

$$\frac{k-1}{k} \leq x < \frac{k}{k+1}.$$

Put

$$\theta_i(x) = \begin{cases} 2^{1-p_i/2} b_{i-2^{p_i-1},\,k}^{(p_i)} \sqrt{k(k+1)} \sin[\pi(p_i - p_k + 1)(k+1)(kx+1-k)], & 1 \leq k \leq 2^{p_i}, \\ 0, & k > 2^{p_i}; \end{cases}$$

$$\theta_i(1) = 0.$$

It is easy to verify that the system $\{\theta_i(x)\}$ is obtained by the construction of the lemma of § 1, where the role of E_j is played by the half-intervals

$$\left[\frac{j-1}{j}, \frac{j}{j+1}\right),$$

the systems $\{\psi_k^{(j)}(x)\}$ are obtained by compressing the trigonometric system, and the matrices

$$A_n = 2^{\frac{1-n}{2}} B_n.$$

Let us note some properties of the functions $\theta_i(x)$:

1) $\{\theta_i(x)\}$ form an orthonormal system, complete in $L[0, 1]$ (and hence, all the more, in $L^2[0, 1]$).
2) $\theta_i(x) \to 0$ $(i \to \infty)$ everywhere on $[0, 1]$.
3) Moreover,

$$\sum_{i=1}^{\infty} |\theta_i(x)|^{2+\varepsilon} < \infty$$

everywhere on $[0, 1]$, for every $\varepsilon > 0$. This property of the system $\{\theta_i(x)\}$ shows the sharpness of Orlicz' s theorem (namely, the exponent in (1) cannot be increased).

4) The series

$$\sum_{i=1}^{\infty} \theta_i(x)$$

converges everywhere on $[0, 1]$.

Property 4) shows that condition (3) is not necessary for almost-everywhere convergence of a series with respect to a complete system. Moreover, it turns out that there exist everywhere convergent orthogonal series with respect to complete systems whose coefficients tend to infinity. The final solution of this question will be given in the next section.

Remark 2. Let us note that complete systems cannot converge to anything except zero. More precisely, if $\{\varphi_n(x)\}$ is an orthonormal system complete in $L^2[0, 1]$, and $\varphi_n(x) \to F(x)$ $(n \to \infty)$ on E, $\mu E > 0$, then $F(x) = 0$ almost everywhere on E. This fact follows from a well-known theorem of V. Ya. Kozlov ([4]).

§ 3. The main theorem

Let us formulate a condition on the coefficients of convergent orthogonal series which is necessary and sufficient in the class of complete systems.

Theorem 1. *If the orthogonal series (2) with respect to a complete system in $L^2[0,1]$ converges on a set of positive measure, then its coefficients satisfy the condition**

$$\sum_{n=1}^{\infty} \frac{1}{c_n^2} = \infty. \tag{4}$$

Theorem 2 (main). *Let an arbitrary sequence c_n satisfying condition (4) be given. Then there exist an orthonormal system $\{\varphi_n(x)\}$, complete in $[0,1]$, and a function $f(x) \in L^p[0,1]$ for all $1 \le p < 2$, such that the series (2) is the Fourier series of the function $f(x)$, converging to it everywhere on $[0,1]$.*

Combining the assertions of Theorems 1 and 2, we obtain:

Condition (4) is a necessary and sufficient condition for a given sequence c_n to be the sequence of coefficients of an orthogonal series (Fourier series) with respect to a complete system, convergent at every point (or on a set of positive measure).

In other words:

(4) expresses the convergence condition for an orthogonal series, necessary and sufficient in the class of complete systems.

Remark 3. From Theorem 2 there follows the existence of everywhere convergent orthogonal series (and even Fourier series from L^p, $p < 2$) with coefficients tending to infinity. In this connection it is curious to note that if the condition of completeness of the system is replaced by the condition of boundedness in the aggregate, then such a result cannot be obtained, for by the Plancherel-Privalov theorem (([5]; ([2]), p. 173) the coefficients c_n in this case must tend to zero.

Let us note that the proof of the theorems of the present section is based on the lemma of § 1 and on the results of § 2.

§ 4. On Fourier coefficients with respect to complete systems

In the preceding section, in particular, a necessary and sufficient ···

* In order not to stipulate separately the case when a finite number of c_n are equal to zero, in condition (4), in the case $c_n = 0$, we put $1/c_n = 1$.

sufficient condition in order that a given sequence c_n be the sequence of coefficients of a convergent Fourier series with respect to a complete system. Let us remove the requirement that the series converge. We obtain the following formulation of the question: **under what conditions are the given numbers c_n the Fourier coefficients of some function with respect to a complete system?**

If the condition of completeness of the system is replaced by the condition of boundedness in the aggregate, then for this question a number of necessary conditions are known, the simplest of which is contained in the well-known theorem of Mercer ([2], p. 181), asserting that the Fourier coefficients of any function $f(x) \in L$, with respect to an orthonormal system bounded in the aggregate, tend to zero. It turns out that, if one considers the analogous question for complete systems, then a fundamentally different result is obtained. Namely, the following theorem is true:

Theorem 3. Let an arbitrary sequence c_n be given. Then there exist a complete orthonormal system $\{\varphi_n(x)\}$ in $L[0,1]$ and a function $f(x) \in L^p$ for all $1 \leq p < 2$ such that c_n are the Fourier coefficients of the function $f(x)$ with respect to the system $\{\varphi_n(x)\}$.

Thus, Theorem 3 shows that nothing at all can be said in advance about the Fourier coefficients of functions in L^p, $p < 2$, with respect to arbitrary complete systems. In particular, they may tend to infinity, and as rapidly as desired.

In the formulation of Theorem 3 the function $f(x)$, just as the system $\{\varphi_n(x)\}$, depends on the sequence c_n. In fact, a stronger version of this theorem is true, in which the function $f(x)$ is arbitrary.

Remark 4. It is interesting to compare Theorem 3 with the results of § 3. It follows from Theorem 3 that any sequence of numbers c_n may serve as the Fourier coefficients of functions in L^p with respect to a complete system. But if one requires convergence of the corresponding Fourier series, even only on a set of positive measure, then, by Theorems 1 and 2, there already arises the necessary and sufficient condition (4).

§ 5. On complete systems bounded in the aggregate. The questions considered in the note, as we have seen, have been fully studied in the class of complete systems. It is clear, however, that analogous formulations of the questions are possible also in other classes, for example in the class of complete systems bounded in the aggregate. In this case it turns out that these problems require essentially different methods of consideration. Let us note some assertions analogous to the results of § 2. Namely, one can prove that there exists a complete orthonormal system $\{\varphi_n(x)\}$, bounded in the aggregate, such that $\varphi_n(x) \to 0$ $(n \to \infty)$ on a set of positive measure (it is obvious that systems satisfying this condition almost everywhere on $[0,1]$ certainly do not exist). Moreover, there exist orthogonal series, convergent on a set of positive measure, with respect to complete systems bounded in the aggregate, with coefficients tending to infinity. However, the question of necessary and sufficient conditions on the coefficients of such series (analogous to condition (4)) remains open.

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Note: Figure translations are in progress. See original paper for figures.