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Abstract

Full Text

MATHEMATICS

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STABLE SITUATIONS IN COALITION GAMES

(Presented by Academician A. N. Kolmogorov on 27 XI 1959)

By coalition games we mean games of such a type in which certain groups (coalitions) of players possess interests inherent in these groups as such. The payoff of a group is understood as a quantitatively measured realization of the goal set before this group, and is not subject to division among the players who are members of this group. (We note that such a characterization of coalition games does not fully correspond to their description given by the author in the survey article ([1]).)

One of the simplest variants of the theory of coalition games (“coalitioners’ games”) was considered in the note ([2]), whose definitions and notation will be assumed known. Let us note at once that the model of every coalitioners’ game can be constructed within the framework of a certain noncoalitional game. Therefore, relying only on Nash’ s theorem ([3]), one can prove (and moreover without using the apparatus of consistent families of measures and Markov measures ([2])) the following strengthening of Theorem 3 of the note ([2]).

Theorem 1. *Every coalition game (“coalitioners’ game”) has mixed equilibrium situations in which the individual strategies of the players are independent. Moreover, one may require that the set K in the definition of an equilibrium situation be empty.*

The purpose of the present note is to present a new variant of the theory of coalition games. All sets of the form S_i ($i \in I$) will be assumed finite, and all subsets S_K ($K \subset I$) measurable for every measure μ_K considered below on S_K .

1. Let \mathfrak{R} be a complex of subsets of the set I . A consistent family of measures $\mu_{\mathfrak{R}}$ on $S_{\mathfrak{R}} = \{S_K\}_{K \in \mathfrak{R}}$ is called **internal** if, for any $K \in \mathfrak{R}$ and $f_K \in S_K$, one has $\mu_K(f_K) > 0$. The set of all internal consistent families of measures on S_K will be denoted by $W(S_{\mathfrak{R}})$.

Theorem 2. *There exists a topological mapping φ of the set $W(S_{\mathfrak{R}})$ into a Euclidean space E of the corresponding dimension such that all conditional probabilities of the form $\mu_{R \cup K}(f_R | f_K)$ turn out to be uniformly continuous, on the whole image $\varphi W(S_{\mathfrak{R}})$, functions of the family $\mu_{\mathfrak{R}}$ in the sense of the metric of E , and the closure $\varphi W(S_{\mathfrak{R}})$ in E is homeomorphic to a closed ball of the corresponding dimension.*

2. Take arbitrary $K \subset L \subset I$ and say that a **random transition** $\pi(L | K)$ is given on S_L if, for any $R \subset L$, $N \subset K$, $x \in S_R$, $y \in S_N$, numbers $\pi_{R,N}^{L,K}(x | y)$ are defined, satisfying the following conditions:

- 1°. For any fixed $N \subset K$ and $y \in S_N$

$$\pi_{R,N}^{L,K}(x | y), \quad R \subset L, \quad x \in S_R,$$

constitutes a consistent family of measures on $\{S_R\}_{R \subset L}$.

- 2°. For any $R \subset L$, $x \in S_R$, $N \subset K$, $y \in S_N$, as soon as $x_{N \cap R} = y_{N \cap R}$,

$$\pi_{R,N}^{L,K}(x | y) = \pi_{R \setminus N, N}^{L,K}(x_{R \setminus N} | y).$$

- 3°. For any $N_1, N_2 \subset K$ for which $N_1 \cap N_2 = \Lambda$, and arbitrary $R \subset L$, $x \in S_R$, $y \in S_N$, one has

$$\pi_{R,N}^{L,K}(x | y) = \sum_{z \in S_{N_2}} \pi_{R, N_1 \cup N_2}^{L,K}(x | y \times z) \pi_{N_1 \cup N_2, N_1}^{N_1 \cup N_2, N_1 \cup N_2}(x \times y | z).$$

Random transitions $\pi'(L_1 | K_1)$ and $\pi''(L_2 | K_2)$ are called **consistent** if, for any $R \subset L_1 \cap L_2$, $N \subset K_1 \cap K_2$, $x \in S_R$, $y \in S_N$,

$$\pi'_{R,N}^{L_1, K_1}(x | y) = \pi''_{R,N}^{L_2, K_2}(x | y).$$

The set of all ordered pairs of subsets I of the form $\langle K, L \rangle$, where $K \subset L \in \mathfrak{R}$, will be denoted by $\Phi_{\mathfrak{R}}$, and the set of all ordered pairs $\langle K, I \rangle$, where $K \in \mathfrak{R}$, by $\Psi_{\mathfrak{R}}$. By $\pi(\Phi_{\mathfrak{R}})$ we shall mean a system of random transitions of the form $\pi(L | K)$, where $\langle K, L \rangle \in \Phi_{\mathfrak{R}}$. The system $\pi(\Psi_{\mathfrak{R}})$ is defined analogously.

A consistent system $\pi(\Psi_{\mathfrak{R}})$ is called an **extension** of the consistent system $\pi(\Phi_{\mathfrak{R}})$ if every random transition from $\pi(\Psi_{\mathfrak{R}})$ is consistent with every random transition from $\pi(\Phi_{\mathfrak{R}})$.

Theorem 3. *In order that every system of the form $\pi(\Phi_{\mathfrak{R}})$ have an extension, it is necessary and sufficient that the complex \mathfrak{R} be regular.*

In what follows, by $\pi(\Psi_{\mathfrak{R}})$ we shall mean some fixed extension of $\pi(\Phi_{\mathfrak{R}})$, called **Markovian**.

3. For each $\mu_{\mathfrak{R}} \in W(S_{\mathfrak{R}})$ (and also for each $\varphi\mu_{\mathfrak{R}} \in \varphi W(S_{\mathfrak{R}})$), the conditional probabilities of the form $\mu_{R \cup K}(f_R | f_K)$ ($R \cup K \in \mathfrak{R}$) determine a certain system of random transitions of the form $\pi(\Phi_{\mathfrak{R}})$. In view of the uniform continuity of these probabilities, the correspondence between points of $\varphi W(S_{\mathfrak{R}})$ and systems of random transitions can, by continuity, be extended to the closure of this set $\overline{\varphi W(S_{\mathfrak{R}})}$. All systems of random transitions obtained in this way will be called **realizable**.

Theorem 4. *The realizability of each system of random transitions is recognized by means of a finite number of elementary operations.*

4. Let I be the set of players; \mathfrak{R} a regular family of coalitions (\mathfrak{R} is a regular complex of subsets of I); S_i , for any $i \in I$, the set of (pure) strategies of player i . The elements S_K ($K \in \mathfrak{R}$) will be called **pure coalition strategies**, and the elements S_I , **pure situations**. Let, further, $\widehat{\mathfrak{R}} \subset \mathfrak{R}$, and let $H_{\widetilde{K}}$ ($\widetilde{K} \in \widehat{\mathfrak{R}}$) be functions taking real values on S . These values are called the **payoffs** of coalition \widetilde{K} in the corresponding situation.

The system

$$\Gamma = \langle I, \mathfrak{R}, \{S_i\}_{i \in \mathfrak{R}}, \widehat{\mathfrak{R}}, \{H_{\widetilde{K}}\}_{\widetilde{K} \in \widehat{\mathfrak{R}}} \rangle \quad (1)$$

will be called a **coalitional game**.

The system

$$\Gamma^* = \langle I, \mathfrak{R}, \{S_i^*\}_{i \in \mathfrak{R}}, \widehat{\mathfrak{R}}, \{H_{\widetilde{K}}^*\}_{\widetilde{K} \in \widehat{\mathfrak{R}}} \rangle$$

is called a **mixed extension** of the game (1) if S_K^* ($K \in \mathfrak{R}$) is the set of all realizable systems of random transitions of the form

$\pi(\Phi_{I_K})$ (I_K denotes the complex of all subsets of the set K). All realizable Markov continuations $\pi(\Psi_{\mathfrak{R}})$ of consistent systems of random transitions of the form $\pi(\Phi_{\mathfrak{R}})$, consisting of systems of the form $\pi(\Phi_{I_K})$ ($K \in \mathfrak{R}$), shall be called **refined situations** Γ^* , and the measures $\pi(\cdot | f_{\Lambda})$ on S_I included in them shall be called **situations**. The payoff function $H_{\widetilde{K}}^*$ in Γ^* is defined as the mathematical expectation

$$H_{\widetilde{K}}^*(\pi) = \sum_{f_I \in S_I} H_{\widetilde{K}}(f_I) \pi(f_I | f_{\Lambda}).$$

If π is some refined situation Γ^* , then for $K \in \mathfrak{R}$ and $f_K \in S_K$, by $\pi \| f_K$ we shall mean the situation (i.e., the measure on S_I) for which, for any $f'_I \in S_I$,

$$(\pi \| f_K)(f'_I) = \pi(f'_I | f_K).$$

Consider some mapping φ of the complex $\widetilde{\mathfrak{R}}$ into \mathfrak{R} , under which $\widetilde{K} \cup \varphi \widetilde{K} \in \mathfrak{R}$ and $\widetilde{K} \cap \varphi \widetilde{K} = \Lambda$. A situation π is called **φ -stable** if, for every $K \in \widetilde{\mathfrak{R}}$ and arbitrary $f_{\widetilde{K}} \in S_{\widetilde{K}}$ and $f_{\varphi \widetilde{K}} \in S_{\varphi \widetilde{K}}$,

$$H_{\widetilde{K}}^*(\pi \| (f_{\widetilde{K}}, f_{\varphi \widetilde{K}})) \leq H_{\widetilde{K}}^*(\pi \| f_{\varphi \widetilde{K}}).$$

If the complex \mathfrak{K} is zero-dimensional, $\tilde{\mathfrak{K}} = \mathfrak{K} \setminus \{\Lambda\}$, and $\varphi\tilde{K} = \Lambda$ for every $\tilde{K} \in \tilde{\mathfrak{K}}$, then the game Γ becomes noncoalitional, and the concept of φ -stability of situations turns into the concept of their equilibrium in the sense of Nash.

Let

$$\mathfrak{K} = \mathfrak{K}_n \supset \mathfrak{K}_{n-1} \supset \dots \supset \mathfrak{K}_1 \supset \mathfrak{K}_0 = \Lambda$$

be a sequence of subcomplexes of \mathfrak{K} , in which \mathfrak{K}_{i-1} ($i = 1, \dots, n$) is a normal subcomplex of \mathfrak{K}_i and is obtained from \mathfrak{K}_i by deleting the stars of all vertices belonging to the set Q_i of all proper vertices of an outer skeleton T_i in \mathfrak{K}_i . Partition each of the sets Q_i into pairwise disjoint sets of players

$$Q_i = K_{i1} \cup K_{i2} \cup \dots \cup K_{ir_i}$$

and take as $\tilde{\mathfrak{K}}$ the family of all sets of the form K_{ij} ($i = 1, \dots, n; j = 1, \dots, r_i$). Put $\varphi^*K_{11} = \Lambda$, and, for $i \neq 1$ or $j \neq 1$, $\varphi^*K_{ij} \subset K_{kl}$, where $k < i$ or $k = i$ and $l < j$. At the same time we shall require that, for any i and j ,

$$K_{ij} \cup \varphi^*K_{ij} \in \mathfrak{K}.$$

Theorem 5. *Every coalitional game Γ , for any mapping of the form φ^* , has φ^* -stable situations in its mixed extension.*

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3. J. Nash, *Ann. of Math.*, **54**, 286 (1951).

Note: Figure translations are in progress. See original paper for figures.

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