RANGE OF VALUES OF THE FUNCTIONAL

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Abstract

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MATHEMATICS

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RANGE OF VALUES OF THE FUNCTIONAL

$$I = \ln \frac{w^\lambda \varphi'(w)^{1-\lambda} \varphi'(0)^\nu}{\varphi(w)^\lambda |\varphi(w)|^\kappa}$$

ON CERTAIN CLASSES OF BOUNDED UNIVALENT FUNCTIONS

(Presented by Academician M. A. Lavrent︠ev on February 26, 1960)

We consider the following classes of functions: S_1 is the class of functions $\varphi(\omega) = \beta\omega + \beta_2\omega^2 + ... + \beta_n\omega^n + ...$, $\beta > 0$, regular, univalent, and satisfying the condition $|\varphi(\omega)| < 1$ in the disk $|\omega| < 1$; $S_1[|\varphi(w)|]$ is the class of functions $\varphi(\omega)$ from S_1 with prescribed $|\varphi(w)|$ at a fixed point w of the disk $|\omega| < 1$; $S_1(\beta)$ is the class of functions $\varphi(\omega)$ from S_1 with prescribed coefficient $\beta = \varphi'(0)$. The following theorems are proved by the method proposed by P. P. Kufarev ([1]).*

Theorem 1. The boundary of the range of values of the functional

$$I = \ln \frac{w^\lambda \varphi'(w)^{1-\lambda} \varphi'(0)^\nu}{\varphi(w)^\lambda}$$

in the class $S_1[|\varphi(w)|]$ is determined by the equation

$$I_0 = \int_{\rho_0}^{r} \left[\lambda \frac{1+q}{1-q} - (1-\lambda) \frac{1-q^2+2q}{(1-q)^2} - \nu \right] \frac{|1-q|^2}{1-\rho^2} \frac{d\rho}{\rho}, \tag{1}$$

where I_0 is a boundary point of the domain; $r = |w|$; $\rho_0 = |\varphi(w)|$; q, $|q| = \rho$, is a root of the equation

$$\overline{B}_0 q - B_0 \overline{q} = (1-\lambda)(1-\rho^2) \left[\frac{qx}{(1-q)^2} - \frac{\overline{q}\,\overline{x}}{(1-\overline{q})^2} \right], \tag{2}$$

in which

$$B_0 = \nu \frac{x + \overline{x}}{2} + (1 - 2\lambda) \frac{x - \overline{x}}{2}, \qquad x = e^{-i\alpha},$$

α is a parameter, $-\pi < \alpha \leq \pi$.

Corollary 1. The boundary of the range of values of the functional

$$I = \ln \frac{w\varphi'(w)}{\varphi(w)}$$

in the class $S_1[|\varphi(w)|]$ is determined by the equation

$$I_0 = e^{i\alpha} \ln \frac{(1-r)(1+\rho_0)}{(1+r)(1-\rho_0)}, \qquad -\pi < \alpha \le \pi.$$

Corollary 2. The boundary of the range of values of the functional

$$I = \ln \varphi'(w)\varphi'(0)$$

* By the logarithm is meant that single-valued branch for which $\operatorname{Im} I \to 0$ as $w \to 0$; λ, ν, κ are arbitrary real numbers, and w is any fixed point of the disk $|w| < 1$.

in the class $S_1[|\varphi(w)|]$ consists of the curve

$$I_0 = \begin{cases} -e^{i\alpha} \ln \dfrac{r^2(1-\rho_0^2)}{\rho_0^2(1-r^2)}, & |\alpha| \leqslant 2 \arcsin \rho_0; \\ \ln \dfrac{4\rho_0^2(1-\rho_0^2)}{\sin^2 \alpha} \pm 4i \left(\dfrac{|\alpha|}{2} - \arcsin \rho_0 \right) - e^{i\alpha} \ln \dfrac{r^2 \operatorname{ctg}^2 \dfrac{\alpha}{2}}{1-r^2}, & 2\arcsin \rho_0 \leqslant |\alpha| \leqslant 2 \arcsin r; \\ \ln \dfrac{\rho_0^2(1-\rho_0^2)}{r^2(1-r^2)} \pm 4i(\arcsin r - \arcsin \rho_0), & 2 \arcsin r \leqslant |\alpha| \leqslant \pi, \end{cases}$$

and of the straight-line segment joining the endpoints of this curve.

Corollary 3. The boundary of the range of values of the functional

$$I = \ln \frac{w^2 \varphi'(w)\varphi'(0)}{\varphi(w)^2}$$

in the class $S_1[|\varphi(w)|]$ is determined by the equation

$$I_0 = e^{i\alpha} \ln \frac{1-r^2}{1-\rho_0^2}, \qquad -\pi < \alpha \leqslant \pi.$$

The results formulated in Corollaries 1, 2, and 3 were obtained earlier by N. A. Lebedev ([2]) by the method of parametric representations.

Corollary 4. The boundary of the range of values of the functional

$$I = \ln \frac{w\varphi'(0)^\nu}{\varphi(w)}$$

in the class $S_1[|\varphi(w)|]$ is determined by the equation

$$I_0 = \ln \left(\frac{1-r^2}{1-\rho_0^2}\right)^\nu \left(\frac{r_0}{\rho_0}\right)^{1-\nu} - \frac{\nu^2\cos\alpha + i\sin\alpha}{\sqrt{\nu^2\cos^2\alpha+\sin^2\alpha}} \ln \frac{(1+r)(1-\rho_0)}{(1-r)(1+\rho_0)}, \qquad -\pi < \alpha \leqslant \pi.$$

Theorem 2. *The boundary of the range of values of the functional*

$$I = \ln \frac{w^\lambda \varphi'(w)^{1-\lambda}\varphi'(0)^\nu}{\varphi(w)|\varphi(w)|^\chi}$$

in the class S_1 is determined by the equation

$$I_0 = \int_{\rho_0}^{r} \left[\lambda\frac{1+q}{1-q} - (1-\lambda)\frac{1-q^2+2q}{(1-q)^2} - \nu\right] \frac{|1-q|^2\,d\rho}{1-\rho^2}\,\frac{1}{\rho} - \ln\rho_0^\chi, \qquad (3)$$

where q, $|q| = \rho$, is a root of equation (2); ρ_0 is either a root of equation (4) satisfying the condition $0 < \rho_0 \leqslant r$, or zero, if equation (4) has no such roots. In the latter case the boundary point does not belong to the set of values of the functional I.

The equation for determining ρ_0 has the form

$$\begin{vmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 \\ 4b_0 & 3b_1 & 2b_2 & b_3 & 0 & 0 & 0 \\ 0 & 4b_0 & 3b_1 & 2b_2 & b_3 & 0 & 0 \\ 0 & 0 & 4b_0 & 3b_1 & 2b_2 & b_3 & 0 \\ 0 & 0 & 0 & 4b_0 & 3b_2 & 2b_2 & b_3 \end{vmatrix} = 0, \qquad (4)$$

where

$$b_0 = (1-2\lambda)\frac{\chi-\bar\chi}{2} + \nu\frac{\chi+\bar\chi}{2}, \qquad b_1 = \left[\chi\left(1-\lambda-\nu-\frac{\chi}{2}\right) + \right.$$

$$\left. +\bar\chi\left(1-2\lambda-\nu-\frac{\chi}{2}\right)\right]\rho_0^2 + \chi\left(-1+2\lambda-\nu+\frac{\chi}{2}\right) + \bar\chi\left(1-\lambda-\nu+\frac{\chi}{2}\right),$$

$$b_2 = (-1+2\lambda+\nu+\kappa)\frac{x+\bar{x}}{2}\rho_0^4+2(\nu+\lambda-1)(x+\bar{x})\rho_0^2+(1-2\lambda+\nu-\kappa)\frac{x+\bar{x}}{2}, \quad b_3 = \rho_0^2 b_1, \quad b_4 = \rho_0^4 b_0.$$

Corollary 1. The boundary of the range of values of the functional

$$I = \ln \frac{w\varphi'(0)^\nu}{\varphi(w)|\varphi(w)|^\kappa}, \qquad \nu \neq \kappa + 1,$$

in the class S_1 is determined by the equation

$$I_0 = \ln\left(\frac{1-r^2}{1-\rho_0^2}\right)^\nu \frac{r^{1-\nu}}{\rho_0^{1-\nu+\kappa}} - \frac{\nu^2\cos\alpha + i\sin\alpha}{\sqrt{\nu^2\cos^2\alpha+\sin^2\alpha}} \ln\frac{(1+r)(1-\rho_0)}{(1-r)(1+\rho_0)},$$

where

$$\rho_0 = \frac{(1+\kappa-\nu)\cos\alpha}{\sqrt{\nu^2\cos^2\alpha+\sin^2\alpha}+\sqrt{(1+\kappa)^2\cos^2\alpha+\sin^2\alpha}},$$

and the parameter α varies so that $0 < \rho_0 \leq r$.

Corollary 2. The boundary of the range of values of the functional

$$I = \ln \frac{w\varphi'(0)^\nu}{\varphi(w)|\varphi(w)|^{\nu-1}}$$

in the class S_1 is determined by the equation

$$I_0 = \ln(1-r^2)^\nu r^{1-\nu} - \frac{\nu^2\cos\alpha + i\sin^2\alpha}{\sqrt{\nu^2\cos^2\alpha+\sin^2\alpha}} \ln\frac{(1+r)}{(1-r)}, \qquad -\pi < \alpha \leq \pi,$$

and the boundary itself does not belong to the set of values of the functional I in the class S_1.

Corollary 3. The boundary of the range of values of the functional

$$I = \ln \frac{w\varphi'(w)}{\varphi(w)|\varphi(w)|^{2\kappa}}, \qquad \kappa \neq 0,$$

in the class S_1 is determined by the equation

$$I_0 = -e^{i\alpha}\ln\frac{(1+r)(1-\rho_0)}{(1-r)(1+\rho_0)} - \ln\rho_0^{2\kappa},$$

where

$$\rho_0 = \frac{2\kappa \cos\alpha}{1 + \sqrt{1 + 4\kappa^2 \cos^2\alpha}},$$

and the parameter α varies so that $0 < \rho_0 \leq r$.

Corollary 4. The boundary of the range of values of the functional

$$I = \ln \frac{w\varphi'(w)}{\varphi(w)}$$

in the class S_1 is determined by the equation

$$I_0 = e^{i\alpha} \ln \frac{1+r}{1-r}, \qquad -\pi < \alpha \leq \pi,$$

and the boundary itself does not belong to the set of values of the functional I.

Corollary 5. The boundary of the range of values of the functional

$$I = \ln \frac{w^2 \varphi'(w)\varphi'(0)}{\varphi(w)^2 |\varphi(w)|^{3\kappa}}, \qquad \kappa \neq 0,$$

in the class S_1 is determined by the equation

$$I_0 = e^{i\alpha} \ln \frac{1-r^2}{1-\rho_0^2} - \ln \rho_0^{3\kappa},$$

where

$$\rho_0 = \sqrt{\frac{-\frac{3}{2}\kappa \cos\alpha}{1 + \frac{3}{2}\kappa \cos\alpha}},$$

and the parameter α varies so that $0 < \rho_0 \leq r$.

Corollary 6. The boundary of the range of values of the functional

$$I = \ln \frac{w^2 \varphi'(w)\varphi'(0)}{\varphi(w)^2}$$

in the class S_1 is determined by the equation

$$I_0 = e^{i\alpha} \ln(1 - r^2), \qquad -\pi < \alpha \leqslant \pi,$$

and the boundary itself does not belong to the set of values of the functional I.

Corollary 7. If $r > 1/\sqrt{2}$, then the boundary of the range of values of the functional

$$I = \ln \varphi'(w)\varphi'(0)$$

in the class S_1 consists of the curve

$$I_0 = \begin{cases} \ln \operatorname{ctg}^2 \alpha \pm \pi i - e^{i\alpha} \ln \dfrac{r^2 \operatorname{ctg}^2 \frac{\alpha}{2}}{1-r^2}, & \dfrac{\pi}{2} < |\alpha| \leqslant 2 \arcsin r, \\ \ln \dfrac{\cos^2 \alpha}{4r^2(1-r^2)} \pm 4i \left(\arcsin r - \dfrac{|\alpha|}{2} + \dfrac{\pi}{4} \right), & 2 \arcsin r \leqslant |\alpha| \leqslant \pi, \end{cases}$$

and the segment of the straight line connecting the points

$$\ln \frac{1}{4r^2(1-r^2)} \pm 4i \left(\arcsin r - \frac{\pi}{4} \right).$$

If $r \leqslant 1/\sqrt{2}$, then the boundary is determined by the equation

$$I_0 = \ln \frac{\cos^2 \alpha}{4r^2(1-r^2)} \pm 4i \left(\arcsin r - \frac{|\alpha|}{2} + \frac{\pi}{4} \right), \qquad \frac{\pi}{2} < |\alpha| \leqslant 2 \arcsin r + \frac{\pi}{2}.$$

Theorem 3. The boundary of the range of values of the functional

$$I = \ln \frac{w^\lambda \varphi'(w)^{1-\lambda}}{\varphi(w)^\lambda |\varphi(w)|^\varkappa}$$

in the class $S_1(\beta)$ is determined by the equation

$$I_0 = \int_{\rho_0}^{r} \left[\lambda \frac{1+q}{1-q} - (1-\lambda) \frac{1-q^2+2q}{(1-q)^2} \right] \frac{|1-q|^2}{1-\rho^2} \frac{d\rho}{\rho} - \ln \rho_0^\varkappa,$$

where q, $|q| = \rho$, is determined from equation (2) for

$$B_0 = (1-2\lambda) \frac{x-\bar{x}}{2} + k,$$

and the modulus ρ of the boundary function and the real constant k are found from the system of two equations:

$$1) \quad \ln\frac{1}{\beta} = \int_{\rho_0}^{r} \frac{|1-q|^2}{1-\rho^2}\,\frac{d\rho}{\rho};$$

2) equation (4), in which instead of $v\dfrac{x+\bar{x}}{2}$ there stands k.

Corollary. The boundary of the range of values of the functional

$$I = \ln\frac{w|\varphi(w)|^{-\varkappa}}{\varphi(w)}$$

in the class $S_1(\beta)$ is determined by the equation

$$I_0 = \ln\frac{r}{\rho_0^{1+\varkappa}} - \frac{i\sin\alpha(1-\rho_0^2)}{(1+\rho_0^2)\sqrt{(1+\varkappa)^2\cos^2\alpha+\sin^2\alpha} - 2\rho_0(1+\varkappa)\cos\alpha}\ln\frac{(1+r)(1-\rho_0)}{(1-r)(1+\rho_0)},$$

where ρ_0 is found from the relation

$$\ln\frac{\rho_0(1-r^2)}{\beta r(1-\rho_0^2)} = \frac{(1+\varkappa)(1+\rho_0^2)\cos\alpha - 2\rho_0\sqrt{(1+\varkappa)^2\cos^2\alpha+\sin^2\alpha}}{(1+\rho_0^2)\sqrt{(1+\varkappa)^2\cos^2\alpha+\sin^2\alpha} - 2\rho_0(1+\varkappa)\cos\alpha}\ln\frac{(1+r)(1-\rho_0)}{(1-r)(1+\rho_0)}.$$

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CITED LITERATURE

1. P. P. Kufarev, DAN, 107, No. 5, 633 (1956).

2. N. A. Lebedev. Vestn. LGU, No. 11, issue 4, 3 (1955).

Note: Figure translations are in progress. See original paper for figures.