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Fig. 1

Figure 1: Fig. 1

Abstract

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COMPLETENESS QUESTIONS FOR SYSTEMS OF AUTOMATA*

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In mathematics and its applications there often arises the problem of the possibility of representing objects of a certain set by means of, for example, a given system of objects of the same set in the form of definite expressions, i.e., the problem of completeness of a system of elements. Thus, in algebra one studies the question of the possibility of representing a function in the form of formulas through functions of some system; in automata theory—the possibility of representing *o*-*d*-operators by means of circuits constructed from elementary automata, and so on. In two-valued logic (P_2) this problem has been solved completely ($^{1-3}$); its solution in many-valued logic has been advanced considerably (^{3,4}), while with respect to automata only the first steps are so far being taken in this direction ($^{5-8}$). In the present note questions of completeness of systems of automata are studied. It turns out that the solution of this problem depends in an essential way on what is understood by a function realized by an automaton, and what is taken as the class of admissible circuits. Therefore the paper gives several definitions of completeness. Some conditions for completeness are presented.

Fig. 1

1°. The concept of an automaton is connected with three concepts: element, circuit, and function**.

I. An element F_i, Fig. ($1a$), has some number r_i of inputs and one output.

Let there be given an alphabet $X = \{x_i\}$.

II. Circuit.

0. A pole with a letter $x_i \in X$ assigned to it is a circuit (Fig. $1b$). It is an input and output pole of the circuit.
1. The union of two circuits S_1 and S_2 is a circuit (S) (Fig. $1c$). The set

of input poles of the circuit S is the union of the sets of input poles of S_1 and S_2. The set of output poles of the circuit S is some subset of the union of the output poles of the circuits S_1 and S_2.

2. The result of attaching the inputs of an element to the output poles of a circuit S is a circuit. In this case it is allowed to attach several inputs of the element to one output of the circuit (Fig. 1d). The inputs of the resulting circuit S are the inputs of the circuit, and the outputs are the output of the element and some subset of the output poles of the original circuit.

As a result of such a construction, Boolean variables $x_1, \ldots, x_n$ will be assigned to the input poles of the circuit.

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* Here by an automaton is meant a "finite automaton," i.e., one having a finite number of states.

** Cf. with S. V. Yablonskii' s definition of a control system (^{9}).

III. 1. To an element there is assigned a pair $(f(\ldots), t)$, where f is an r-ary Boolean operator (^{3}), and t is a nonnegative integer. This means that if, at time τ, the inputs of the element to which the pair $(f(\ldots), t)$ corresponds are in states $(\alpha_1, \ldots, \alpha_r)$, then its output at time $\tau + t$ is in the state $f(\alpha_1, \ldots, \alpha_r)$. Thus, the element F_i processes information with delay t. An element with an assigned pair will be called an elementary automaton.

2. Obviously, knowing the nature of the processing of information by each element, i.e., the pair (f, t), and the structure of the circuit, determined by an inductive process, we can in some cases naturally associate pairs with the nodes of the circuit (including the outputs) in such a way that these pairs will characterize the processing of information by the corresponding part of the circuit. Such a definition is always possible if the condition is fulfilled that information is supplied simultaneously to all inputs of each element of the circuit.

A path in a circuit is called a sequence

$$a_1 b_1 a_2 b_2 a_3 \ldots a_s b_s a_{s+1}, \qquad (1)$$

where b_i is an elementary automaton, a_i is its input, and a_{i+1} is its output. The length of the path (1) is the number $\sum_{i=1}^{s} t_i$, where t_i is the delay of the elementary automaton b_i.

At the output B of the circuit, by definition, a pair $(F_B(\ldots), T)$ is realized, where $F_B(x_1, \ldots, x_n)$ is such a function that $F_B(\alpha_1, \ldots, \alpha_n)$ is, for any tuple at time τ on the inputs $(\alpha_1, \ldots, \alpha_n)$, the state of the output B at time $\tau + T$, where T is the length of an arbitrary path from an input to the output B.

In sections 1°–3° we shall consider such circuits that the lengths of all paths to an arbitrary fixed element from all inputs of the circuit connected with it are equal. A circuit all of whose elements belong to the set $\mathfrak{F}$ will be called a circuit over $\mathfrak{F}$. In what follows we shall sometimes denote a circuit, in particular an elementary automaton, realizing the pair (f, t), by $[f, t]$. It is easy to see that, by assigning certain states to the outputs of a circuit at the initial moments, we obtain a finite automaton.

2°. **Definition 1.** A system $\mathfrak{F}$ of elementary automata $F_i = [f_i, t_i]$ will be called **complete in the first sense** if, for any function $f(x_1, \dots, x_p)$ from P_2 and any delay t, one can construct a circuit over $\mathfrak{F}$ realizing the function $f(x_1, \dots, x_p)$ with delay t.

Theorem 1. A system $\mathfrak{F}$ is complete in the first sense if and only if it contains: a) some system of elementary automata $[f_i, 0]$ such that $\{f_i\}$ forms a complete system in P_2; b) some elementary automaton $[f, 1]$, where f is not a constant.

For example, a system complete in the first sense is the system $[x, 1]$, $[f_1, 0], \dots, [f_s, 0]$, where $f_1, \dots, f_s$ is a complete system of functions in P_2.

Proof of the theorem. Sufficiency. By virtue of b), in the system $\mathfrak{F}$ there is an elementary automaton $[f_a, 1]$ such that $f_a(x_1, \dots, x_n)$ is not a constant; therefore one can find a pair of neighboring tuples on which $f_a(x_1, \dots, x_n)$ takes different values. But then, in an immediately obvious way, one can construct a circuit with a single output realizing $(x, 1)$. Consequently, any pair (φ, t) is also realizable.

Necessity. If in $\mathfrak{F}$ there is no pair with $t = 1$, then, obviously, no pair $(\varphi, 1)$ can be realized. If $\mathfrak{F}$ contains only a pair $(\varphi, 1)$, where φ is a constant, then, obviously, the pair $(x, 1)$ cannot be realized.

Corollary. There does not exist a single element forming a complete system in the first sense.

3°. **Definition 2.** A system $\mathfrak{F}$ is **complete in the second sense** if, for any function $f(x_1, \dots, x_n)$ from P_2, there exist t and a circuit over $\mathfrak{F}$

such that this circuit realizes the pair (f, t). Obviously, every system complete in the first sense is also complete in the second.

Theorem 2. *There is no single elementary automaton* $[\varphi, t]$ *with* $t > 0$ *such that it forms a complete system in the second sense of completeness.*

Proof. Suppose that such an elementary automaton $[\varphi, t]$ exists. Then φ must obviously be a Sheffer function, and $\varphi(0, \dots, 0) = 1$, $\varphi(1, \dots, 1) = 0$. Consider an arbitrary circuit over $\mathfrak{F} = \{(\varphi, t)\}$. Identify its inputs. Then it is easy to show by induction (using equality of the lengths of the paths from the input to an arbitrary elementary automaton) that every elementary automaton realizes either x or $\bar{x}$, i.e. there is no circuit realizing, for example, the pair $(0, t)$. The theorem is proved.

Lemma 1. *The system* $\{[x,t],[f_i,t_i]\}$, *where* $\{f_i\}$ *is complete in* P_2 *and the* t_i *are multiples of* t, *is complete in the second sense.*

Corollary. If, from a system of elementary automata, one can construct circuits realizing pairs satisfying the condition of Lemma 1, then it is complete in the second sense.

Lemma 2. *From the system* $\{[\theta,t_1],[c,t_2]\}$, *where*

$$\theta(x_1,x_2,x_3)=\bar{x}_1x_2x_3\vee x_1\bar{x}_3\vee\bar{x}_1\bar{x}_2\bar{x}_3,$$

$c(x)$ *is a constant (the element* $[c,t_2]$ *has one input), one can construct circuits realizing* (x,T) *and* (θ,lT), *where* $T\geq 0$, l *is a nonnegative integer, for any* t_1,t_2.

Proof. It is obvious that $\theta(x_1,x_2,x_2)\equiv\bar{x}_1\vee\bar{x}_2$. Therefore the case $t_1=0$ is trivial. Consider the case $t_1>0$. By attaching to all inputs of the elementary automaton $[\theta,t_1]$ one element $[c,t_2]$ each, we obtain $[c,t_1+t_2]$. From $[c,t_2]$ and $[c,t_1+t_2]$ one can (by attaching them to each other in different orders) obtain $[0,t_1+2t_2]$ and $[1,t_1+2t_2]$. Therefore, for any positive integers a and b, one can realize both constants 0 and 1 with delays $at_2+b(t_1+t_2)$. Further, from the elementary automaton $[\theta,t_1]$ one can obtain $[x,2kt_1]$ for any positive integer k. Since $\theta(0,1,x)=x$, under the condition

$$2kt_1=at_2+b(t_1+t_2),\tag{2}$$

by attaching to the inputs x_1,x_2,x_3 of the elementary automaton $[\theta,t_1]$, respectively, $[0,at_2+b(t_1+t_2)]$, $[1,at_2+b(t_1+t_2)]$, and $[x,2kt_1]$, we obtain $[x,(2k+1)t_1]$. Equation (2) has, for example, the solution $k=3t_2+t_1$, $b=2t_1$, $a=4t_1$, and for $t_1>0$ the numbers k,b,a are positive. Thus we have $[\theta,t_1]$ and $[x,(2k+1)t_1]$. Attaching to each input of the elementary automaton $[\theta,t_1]$, according to the circuit $[x,(2k+1)t_1]$, we obtain $[\theta,2(k+1)t_1]$. In addition, we already have $[x,2t_1]$. The lemma is proved.

Corollary. There exists a pair of elementary automata $[\psi_1,t_1]$ and $[\psi_2,t_2]$ complete in the second sense for any t_1 and t_2.

Theorem 3. *The conditions of the corollary to Lemma 1 are necessary and sufficient for the system* $\{[f_i,t_i]\}$ *to be complete in the second sense.*

Proof. Sufficiency—see the corollary to Lemma 1. **Necessity.** Let the system $\{[f_i,t_i]\}$ be complete in the second sense. Then from it one can, in particular, construct $[\theta,\tau_1]$ and $[c,\tau_2]$, where

$$\theta(x_1,x_2,x_3)=\bar{x}_1x_2x_3\vee x_1\bar{x}_3\vee\bar{x}_1\bar{x}_2\bar{x}_3.$$

By Lemma 2 one can construct $[\theta,lT]$ and $[x,T]$. The theorem is proved.

4°. We shall now consider arbitrary *o*-*d*-operators (5^{-1}), i.e. transformations of sequences $x(i)$ into sequences $z(i)$, defined by the system of equations

$$q(t)=\psi[x(t),q(t-1)],\qquad z(t)=\varphi[x(t),q(t-1)],\qquad q(0)=c,\tag{3}$$

Fig. 2

Figure 2: Fig. 2

where each of the parameters x, q, z ranges over its own finite set,

values (finite automata) (see Fig. 2). Not every o-d operator can be realized by an automaton defined in 1°. Let us extend the definition of an automaton.

I. An element has some number r of inputs and some number m of outputs.

II. Circuit. Items 0, 1, and 2 from 1° are retained; in addition:

3. The result of adjoining an input a' of a circuit S to an output of the circuit is a circuit (S'). The inputs of the circuit S' are the inputs of the circuit S except for a', and the outputs are certain outputs of the circuit S.

Fig. 2

III. The function (o-d operator) of a circuit is defined in the usual way ($^{5-7}$). To an element there is put in correspondence an o-d operator with input alphabet consisting of 2^r words and output alphabet consisting of 2^m words. An element together with the o-d operator corresponding to it will be called an elementary automaton.

Moreover, the circuits must satisfy the "consistency" conditions (^{5,7}). In each particular case these conditions are easily checked and, in essence, mean that for every state of the circuit, along every directed cycle without delays, an identical transformation of information takes place. Mathematically, this restriction means that the process of computing the operator realized by the circuit is uniquely executable. As is known (^{5,7}), in order to realize all o-d operators it suffices to have a complete system of elements with zero delay and unit delay.

It follows from Theorem 2 that there does not exist a single elementary automaton (of the kind considered in 1°) forming a complete system.

Theorem 4. *There exists a single elementary automaton (an analogue of the Sheffer function in P_2), by copies of which one can realize any o-d operator.*

Let us give an example of such an elementary operator. Consider the function

$$f(x_1, x_2, x_3, x_4) = \bar{x}_1 x_4 \vee \bar{x}_2 \bar{x}_4 \vee \bar{x}_3.$$

The elementary automaton F of interest to us is obtained from the element realizing the function f with zero delay by adjoining a unit delay to the input x_4 (see Fig. 3). Indeed, when the inputs x_1 and x_2 in the elementary automaton F are identified, we obtain $[\varphi, 0]$, where $\varphi = \bar{x}_1 \vee \bar{x}_3$, and when the constant 1 is substituted for x_3 and x_2, the constant 0 for x_1, and the inputs of the resulting circuit are identified, we obtain a unit delay.

Fig. 3

Figure 3: Fig. 3

Fig. 3

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References

1. E. Post, *Am. J. Math.*, **43**, 163 (1921).
2. S. V. Yablonskii, *Matem. sbornik*, **30** (72), No. 2, 328 (1952).
3. S. V. Yablonskii, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **5** (1958).
4. Yu. I. Yanov, A. A. Muchnik, *DAN*, **127**, No. 1, 44 (1959).
5. A. W. Burks, J. B. Wright, *Proc. IRE*, **41**, No. 10, 1357 (1953).
6. B. A. Trakhtenbrot, *DAN*, **112**, No. 6, 1005 (1957).
7. N. E. Kobrinskii and B. A. Trakhtenbrot, collection *Logical Investigations*, Academy of Sciences of the USSR Press, 1959, p. 352.
8. M. L. Minskii, collection *Automata*, IL, 1956, p. 163.
9. S. V. Yablonskii, collection *Problems of Cybernetics*, issue 2, 1959.

Note: Figure translations are in progress. See original paper for figures.

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