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Abstract

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ESTIMATES OF THE RESOLVENTS OF SINGULAR ELLIPTIC OPERATORS

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In the present note, estimates are given for the resolvents of one class of elliptic operators, and with their help theorems on the completeness of the eigenfunctions and associated functions of these operators are established.

1°. Consider an operator L in the space of N-dimensional vector-functions

$$L = (-1)^{m+1} \sum_{k_1+\cdots+k_n=2m} A^{k_1\dots k_n}(x) \frac{\partial^{2m}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} + L_1\left(x, \frac{\partial}{\partial x}\right) + Q(x), \quad \text{(a)}$$

where $x = (x_1, \dots, x_n)$, $-\infty < x_k < \infty$, $k = 1, 2, \dots, n$; $A^{k_1,\dots,k_n}(x)$ is the matrix $\|a_{ij}^{k_1,\dots,k_n}(x)\|$, $i, j = 1, 2, \dots, N$. By $L_1\left(x, \frac{\partial}{\partial x}\right)$ is denoted a linear differential operator of order $< 2m$; $Q(x)$ is the operator of multiplication by the matrix $Q(x)$.

We shall assume that the following conditions are fulfilled:

a) The characteristic roots $\lambda_i(s, x)$ of the matrix

$$(-1)^{m+1} \sum_{k_1+\cdots+k_n=2m} A^{k_1\dots k_n}(x)(is_1)^{k_1} \dots (is_n)^{k_n}$$

for real s and $|s|^2 = \sum_{k=1}^n s_k^2 = 1$ and any $x \in R_n$ satisfy the inequalities $\operatorname{Re} \lambda_i(s, x) < -\delta$, where $\delta > 0$ and does not depend on s and x.

b) The matrix $Q(x) = \|q_{ij}(x)\|$ is symmetric, and its characteristic roots $\beta_k(x)$ are such that $\beta_k(x) \leq -g(x)$, while $g(x)$ satisfies the inequality $g(x) \geq c|x|^\alpha + c_1$ for some $c > 0$, $\alpha > 0$.

Let us note that if the matrices $A^{k_1\dots k_n}(x)$ are symmetric, then the operator L will be strongly elliptic.

Assume further that the elements of the matrices $A^{k_1\dots k_n}(x)$ and $Q(x)$ have $2m$ continuous derivatives, and that all derivatives of the elements of the matrices $A^{k_1\dots k_n}(x)$ are bounded in the whole space R_n, while the k-th derivative $q_{ij}^{(k)}(x)$ of the element $q_{ij}(x)$ of the matrix $Q(x)$ satisfies the inequality

$$|q_{ij}^{(k)}(x)| \leq c g^{\frac{2m+k}{2m}}(x), \qquad i, j = 1, \dots, N. \tag{1}$$

We shall study the principal part of the operator L:

$$L_0 = (-1)^{m+1} \sum_{k_1+\dots+k_n=2m} A^{k_1\dots k_n}(x) \frac{\partial^{2m}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} + Q(x). \tag{2}$$

Theorem 1. *If the operator (2) satisfies conditions a), b), and also inequality (1), then there exists a number $\lambda_0 > 0$ such that the operator*

$(L_0 - \lambda_0 E)^{-1}$ is an integral operator whose kernel $H(x, y)$ satisfies the estimate

$$\left| \frac{\partial^k H(x,y)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right| \leq \frac{B_\varepsilon e^{-c|x-y|}}{g^{\frac{2m-k-\varepsilon}{2m}}(x)}, \qquad \sum_{j=1}^{n} k_j = k, \quad k = 0, \dots, 2m-1. \tag{3}$$

Here ε is any number > 0; $|x - y| \geq 1$; $c > 0$; B_ε is a constant depending on ε. In a neighborhood of the point $x = y$, the kernel $H(x, y)$ has the singularity of the fundamental solution.

The following simple lemmas hold.

Lemma 1. If $g(x) \geq c|x|^\alpha + c_1$, then there exists an integer m such that the operator $(L_0 - \lambda_0 E)^{-m}$ will be an operator of Hilbert—Schmidt type.

Lemma 2. If

$$|M(x)| \leq c g^{\frac{2m-k}{k} - \delta}(x)$$

for some $\delta > 0$, then

$$M(x) \frac{\partial^k}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} (L_0 - \lambda_0 E)^{-1}, \qquad k_1 + \dots + k_n = k \leq 2m - 1,$$

is a completely continuous operator.

2°. Theorem 1 and the lemmas obtained on its basis make it possible to draw a conclusion about the completeness of the eigenfunctions and associated functions, using the well-known theorem of M. V. Keldysh (^{1}).

Consider the operator

$$L = (-1)^{m+1} \sum_{k_1+\dots+k_n=2m} A^{k_1\dots k_n}(x) + L_1'\left(x, \frac{\partial}{\partial x}\right) + Q(x) = L_0 + L_1',$$

which is formally symmetric as a whole. Let the coefficients of the operator L_1', whose order is $< 2m$, be bounded functions throughout R_n.

Theorem $1'$**.** If the coefficients of the operator L_1' are bounded in the whole space R_n, then there exists a number λ_0 such that $(L-\lambda_0 E)^{-1}$ is an integral operator with kernel $H_1(x,y)$, which satisfies inequality (3).

Theorem 2. Let the matrices $A^{k_1\dots k_n}(x)$ be symmetric and let conditions a), b) be fulfilled, and let the elements of the matrix $Q(x)$ satisfy inequality (1). If, for the complex-valued coefficients $b_{ij,l}(x)$ of the operator $B_l\left(x, \dfrac{\partial}{\partial x}\right)$

$$B_l\left(x, \frac{\partial}{\partial x}\right) = \sum_{m_1+\dots+m_n=l} B^{m_1\dots m_n}(x)\frac{\partial^l}{\partial x_1^{m_1}\dots\partial x_n^{m_n}}$$

for some $\delta > 0$ the inequalities

$$|b_{ij,l}(x)| \le c g^{\frac{2m-l}{2m}-\delta}(x), \qquad l = 0, 1, \dots, 2m-1, \tag{4}$$

are satisfied, then the operator

$$\overline{L} = (-1)^{m+1}\sum_{k_1+\dots+k_n=2m} A^{k_1\dots k_n}(x)\frac{\partial^{2m}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}} + \sum_{l=0}^{2m-1} B_l\left(x, \frac{\partial}{\partial x}\right) + Q(x)$$

has a complete system of eigenfunctions and associated functions.

Theorem 2 follows directly from Theorem $1'$ and the lemmas of item 1. Indeed, the operator $\overline{L}$ can be written in the form

$$\overline{L} = (-1)^{m+1}\sum_{k_1+\dots+k_n=2m} A^{k_1\dots k_n}(x) + L'\left(x, \frac{\partial}{\partial x}\right) + \sum_{l=0}^{2m-1} B_l'\left(x, \frac{\partial}{\partial x}\right) + Q(x),$$

where the operator L_0', equal to

$$L_0' = (-1)^{m+1}\sum_{k_1+\dots+k_n=2m} A^{k_1\dots k_n}(x)\frac{\partial^{2m}}{\partial x_1^{k_1}\dots\partial x_n^{k_n}} + L'\left(x, \frac{\partial}{\partial x}\right) + Q(x),$$

will be a self-adjoint operator and condition (3) is satisfied for it. The coefficients of the operators $B_l'\left(x, \frac{\partial}{\partial x}\right)$ will, as before, satisfy inequality (4).

Let us now rewrite the equality

$$\overline{L}u = \lambda u$$

in the form*

$$u + \sum_{l=0}^{2m-1} B_l' L_0'^{-1} u = \lambda L_0'^{-1} u.$$

By virtue of Lemma 1, some power of the operator $L_0'^{-1}$ is a Hilbert-Schmidt operator, while the operator $\sum_{l=0}^{2m-1} B_l' L_0'^{-1}$ is completely continuous. From the theorem of M. V. Keldysh we obtain the required assertion.

3°. Let us briefly outline the proof of the main theorem 1. Together with the elliptic operator (α), consider the parabolic operator

$$P = \frac{\partial}{\partial t} - (-1)^{m+1} \sum_{k_1+\dots+k_n=2m} A^{k_1+\dots+k_n}(x) \frac{\partial^{2m}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} - Q(x). \qquad (\beta)$$

The Green function $G(t,\tau,x,y)$ of this operator, as is known $({}^{2,3})$, can be sought in the form

$$G(t,\tau,x,y) = \overline{G}(t,\tau,x-y,x) + \int_\tau^t d\mu \int_{-\infty}^{\infty} \overline{G}(t,\mu,x-\xi,x)\varphi(\mu,\tau,\xi,y)\, d\xi.$$

The matrix $\varphi(t,\tau,x,y)$ satisfies the integral equation

$$K(t,\tau,x-y,x)+\varphi(t,\tau,x,y)+\int_\tau^t d\mu \int_{-\infty}^{\infty} K(t,\mu,x-\xi,x)\varphi(\mu,\tau,\xi,y)\, d\xi = 0. \quad (5)$$

The kernel $K(t,\tau,x-y,x)$ is defined by the equality

$$K(t,\tau,x-y,x) =$$

$$= \left(\frac{\partial}{\partial t} - (-1)^{m+1} \sum A^{k_1 \dots k_n}(x) \frac{\partial^{2m}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} + Q(x) \right) \overline{G}(t,\tau,x-y,x).$$

The function $\overline{G}(t,\tau,x-y,x)$ is obtained in the following way. One constructs the Green function $\overline{G}(t,\tau,x-y,z)$ of the operator L_z with constant coefficients and parametric point z:

$$L_z = \frac{\partial}{\partial t} - (-1)^{m+1} \sum_{k_1+\dots+k_n} A^{k_1\dots k_n}(z) \frac{\partial^{2m}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} + Q(z).$$

Then, putting $z = x$, we obtain $\overline{G}(t, \tau, x - y, x)$.

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* Obviously, without loss of generality one may assume $\lambda_0 = 0$.

It is not difficult to show that, under the assumptions made on the coefficients of the operators, the inequalities

$$\overline{G}(t, \tau, x - y, x) \leq \frac{c_1 \exp\left(-c\,\frac{|x-y|^{2m'}}{(t-\tau)^{1/(2m-1)}} - Bg(x)(t-\tau)\right)}{(t-\tau)^{n/2m}},$$

$$|K(t, \tau, x-y, x)| \leq \frac{c_2}{(t-\tau)^{(n+2m-1+\varepsilon)/2m}} \exp\left(-c_3 \frac{|x-y|^{2m'}}{(t-\tau)^{1/(2m-1)}} - B_1 g(x)(t-\tau)\right).$$

hold. Here all constants are positive and depend neither on x nor on t, $1 > \varepsilon \geq 0$ and $1/2m + 1/2m' = 1$.

Solving equation (5) by the method of successive approximations, one can obtain the estimate:

$$|G(t, 0, x, y)| \leq \frac{c_1}{t^{n/2m}} \exp\left(-c\frac{|x-y|^{2m'}}{t^{1/(2m-1)}} - g(x)t + Mt\right) + \frac{c_\delta \exp\left(-\frac{c_2|x-y|^{2m'}}{t^{1/(2m-1)}} + Mt\right)}{g^{1-\delta}(x) t^{(n-\varepsilon_1)/2m}}, \tag{6}$$

where $\varepsilon_1 > 0$ and δ is any number > 0.

Integrating inequality (6) with respect to t from 0 to ∞, we obtain the required estimate for $H(x, y)$. Of course, one must first shift away from the spectrum, if necessary. After this the constant M in the inequality can be made negative and the integration becomes possible. For the derivatives of the Green’ s function of the parabolic operator, analogous estimates are obtained, and inequality (3) follows from them.

4°. One can somewhat broaden the class of operators for which resolvent estimates hold, making it possible to draw conclusions about the completeness of root vectors on the basis of M. V. Keldysh’ s theorem; however, the formulations of the theorems become somewhat more complicated. We give such an example.

Consider an operator L of the form

$$L = (-1)^{m_1} \frac{\partial^{2m_1}}{\partial x_1^{2m_1}} + \dots + (-1)^{m_n} \frac{\partial^{2m_n}}{\partial x_n^{2m_n}} + (q(x) + ip(x)). \tag{7}$$

Assume that $q(x)$ has m derivatives ($m = \max\{2m_1, \dots, 2m_n\}$), which satisfy the inequality

$$|q^{(l)}(x)| \le cq^{(l+m)/m}(x), \qquad q(x) \ge c|x|^\alpha + c_2 \tag{8}$$

for some $\alpha > 0$.

Theorem 3. Suppose that

$$\frac{|p(x)|}{q(x)} \to 0, \qquad \text{as } |x| \to \infty,$$

and that conditions (8) are fulfilled. Then operator (7) has a discrete spectrum, and the system of eigenfunctions and associated functions is complete.

We note that the completeness theorem for Sturm–Liouville and Schrödinger operators, under somewhat different assumptions, was proved in (^{4},^{5}).

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Note: Figure translations are in progress. See original paper for figures.

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