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Abstract

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ON THE GROWTH OF ENTIRE FUNCTIONS ADMITTING CERTAIN LOWER ESTIMATES

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The present note is devoted to the proof of a theorem on entire functions, which finds application in the spectral theory of operators and, as it seems to us, is also of independent interest.

We shall denote by z and ζ complex variables, $z = re^{i\varphi} = x + iy$, $\zeta = \rho e^{i\vartheta}$; by C a positive quantity independent of the variables denoted by the letters $r, \varphi, \rho, \vartheta, R, \psi, x, y$.

Theorem. If an entire function $f(z)$ admits the lower estimate

$$|f(z)| \geqslant \exp\{-Cr^\alpha |\operatorname{cosec}\varphi|^k\} \quad (\alpha > 1,\ k \geqslant 0),$$

then the growth of $f(z)$ is not higher than order α and normal type.

The proof consists of two parts. In the first part we establish for $f(z)$ the upper estimate

$$|f(z)| \leqslant C\exp\{Cr^\alpha |\operatorname{cosec}\varphi|^l\} \quad (l \geqslant 0), \tag{1}$$

and in the second part we show (and this is the essential point in our constructions) that the assertion of the theorem follows from this estimate.

1. It is obvious that $f(z)$ can have only real zeros, and moreover $f(0) \neq 0$. Without loss of generality one may assume that $f(z) \neq 0$ for $|z| \leqslant 1$. Choose numbers β and ψ so that $1 < \beta < \alpha$, $0 < \psi \leqslant \pi(1 - \beta^{-1})$, and consider the function

$$F_{\beta,\psi}(\zeta) = f(\zeta^{\beta^{-1}} e^{i\psi/2}). \tag{2}$$

This function is holomorphic and nonzero in the closed upper half-plane; for it the estimate

$$|F_{\beta,\psi}(\zeta)| \geqslant \exp\{-C\rho^{\alpha/\beta} \operatorname{cosec}^k(\vartheta/\beta + \psi/2)\} \geqslant \exp\{-C\rho^{\alpha/\beta} \operatorname{cosec}^k \tfrac{1}{2}\psi\}. \tag{3}$$

holds. By the well-known Carleman formula there is the relation

$$\frac{1}{2\pi}\int_1^R\left(\frac{1}{t^2}-\frac{1}{R^2}\right)\left(\ln_+|F_{\beta,\psi}(t)|+\ln_+|F_{\beta,\psi}(-t)|\right)\,dt+$$

$$+\frac{1}{\pi R}\int_0^\pi \ln_+|F_{\beta,\psi}(Re^{i\theta})|\sin\theta\,d\theta=$$

$$=\frac{1}{2\pi}\int_1^R\left(\frac{1}{t^2}-\frac{1}{R^2}\right)\left(\ln_-|F_{\beta,\psi}(t)|+\ln_-|F_{\beta,\psi}(-t)|\right)\,dt+$$

$$+\frac{1}{\pi R}\int_0^\pi \ln_-|F_{\beta,\psi}(Re^{i\theta})|\sin\theta\,d\theta-\operatorname{Im}\frac{1}{2\pi}\int_0^\pi \ln F_{\beta,\psi}(e^{i\theta})\left(\frac{e^{i\theta}}{R^2}-e^{-i\theta}\right)\,d\theta\quad(R\geqslant 1).$$

It is easy to see that the last term on the right-hand side is bounded in modulus uniformly with respect to ψ and R. Applying relation (3) to estimate the other two terms, we obtain:

$$\frac{1}{\pi R}\int_0^\pi \ln_+\left|F_{\beta,\psi}(Re^{i\theta})\right|\sin\theta\,d\theta\le C\left(\operatorname{cosec}\frac{\psi}{2}\right)^k R^{\alpha/\beta-1},\tag{4}$$

$$\frac{1}{2\pi}\int_1^R\left(\ln_+\left|F_{\beta,\psi}(t)\right|+\ln_+\left|F_{\beta,\psi}(-t)\right|\right)t^{-2}\,dt\le$$

$$\le\frac{4}{3}\frac{1}{2\pi}\int_1^R\left(\frac{1}{t^2}-\frac{1}{(2R)^2}\right)\left(\ln_+\left|F_{\beta,\psi}(t)\right|+\ln_+\left|F_{\beta,\psi}(-t)\right|\right)\,dt\le\tag{5}$$

$$\le\frac{4}{3}\frac{1}{2\pi}\int_1^{2R}\left(\frac{1}{t^2}-\frac{1}{(2R)^2}\right)\left(\ln_+\left|F_{\beta,\psi}(t)\right|+\ln_+\left|F_{\beta,\psi}(-t)\right|\right)\,dt\le C\left(\operatorname{cosec}\frac{\psi}{2}\right)^k R^{\alpha/\beta}.$$

Using the expression for Green' s function for the half-disc $\{\operatorname{Im}\zeta>0,\ |\zeta|<R\}$, as well as Green' s formula, it is not hard to obtain, for the harmonic function $\ln|F_{\beta,\psi}(\zeta)|$, the representation ([1])

$$\ln|F_{\beta,\psi}(\zeta)|=\frac{1}{\pi}\int_{-R}^R \ln|F_{\beta,\psi}(t)|\left\{\frac{\rho\sin\vartheta}{\rho^2+t^2-2\rho t\cos\vartheta}-\frac{R^2\rho\sin\vartheta}{R^4+\rho^2t^2-2R^2\rho t\cos\vartheta}\right\}dt+$$

$$+\frac{1}{2\pi}\int_0^{\pi}\ln|F_{\beta,\psi}(Re^{i\theta})|\left\{\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta-\vartheta)}-\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\theta+\vartheta)}\right\}d\theta. \tag{6}$$

Since both integral kernels are positive, the equality sign in (6) can be replaced only by the sign $\le$, if on the right-hand side $\ln_+$ is written instead of ln. Putting $R = 2\rho$, $\rho \ge 2$, and observing that the first kernel

$$\frac{\rho\sin\vartheta}{\rho^2+t^2-2\rho t\cos\vartheta}\le\begin{cases}\rho t^{-2}\operatorname{cosec}\vartheta, & \text{if } |t|\ge 1,\\ 4\rho^{-1}\sin\vartheta, & \text{if } |t|<1,\end{cases}$$

and the second

$$\frac{(R^2-\rho^2)\,4R\rho\sin\vartheta\sin\theta}{(R^2+\rho^2-2R\rho\cos(\theta-\vartheta))(R^2+\rho^2-2R\rho\cos(\theta+\vartheta))}\le$$

$$\le\frac{4(R+\rho)R\rho}{(R-\rho)^3}\sin\theta=24\sin\theta,$$

we arrive at the relation

$$\ln|F_{\beta,\psi}(\zeta)|\le\frac{\rho}{\pi\sin\vartheta}\int_1^{2\rho}\left(\ln_+\left|F_{\beta,\psi}(t)\right|+\ln_+\left|F_{\beta,\psi}(-t)\right|\right)t^{-2}\,dt+$$

$$+\frac{12}{\pi}\int_0^{\pi}\ln_+\left|F_{\beta,\psi}(2\rho e^{i\theta})\right|\sin\theta\,d\theta+\frac{4\sin\vartheta}{\pi\rho}\int_{-1}^{2}\ln_+\left|F_{\beta,\psi}(t)\right|\,dt.$$

Estimating the first two integrals on the right-hand side by means of inequalities (4) and (5), and observing that the third integral is bounded uniformly with respect to ψ and ρ, we obtain

$$\ln|F_{\beta,\psi}(\zeta)|\le C\rho^{\alpha/\beta}\left(\operatorname{cosec}(\psi/2)\right)^k\operatorname{cosec}\vartheta. \tag{7}$$

Put in this relation $\zeta = r^\beta e^{i\beta\psi/2}$. Then, by virtue of (2), we shall have

$$\ln|f(re^{i\psi})|\le Cr^\alpha(\operatorname{cosec}(\psi/2))^k\operatorname{cosec}(\beta\psi/2)\le Cr^\alpha(\operatorname{cosec}\psi)^{k+1}. \tag{8}$$

The validity of this estimate has so far been proved by us under the condition $0<\psi\le\pi(1-\beta^{-1})$. In an analogous manner one can prove its validity

and for $0>\psi\ge-\pi(1-\beta^{-1})$. Passing now from $f(z)$ to $f(-z)$, we obtain that (8) holds also in the angle $|\pi-\varphi|\le\pi(1-\beta^{-1})$.

In order to obtain an estimate for $f(z)$ in the angle $\pi(1-\beta^{-1})<\varphi<\pi\beta^{-1}$ (for $\beta<2$), put in (7) $\psi=\pi(1-\beta^{-1})$, $\zeta=r^{\beta}e^{i\beta(\varphi-\psi/2)}$. We shall have

$$\ln|f(re^{i\varphi})|\le Cr^{\alpha}+C.$$

An analogous estimate obviously holds also for $-\pi\beta^{-1}<\varphi<-\pi(1-\beta^{-1})$. Combining the estimates obtained, we arrive at the conclusion that for $1\le r<\infty$, $0\le\varphi<2\pi$, (1) holds with $l=k+1$.

2. Denote by L the rhombus with vertices at the points $\pm a=\pm 2\operatorname{cosec}(\pi/8l)$, $\pm b=\pm 2i\sec(\pi/8l)$, and by Γ its boundary. Consider the function $\mu(z)=\exp\{-(a^2-z^2)^{-2l}\}$. This function is holomorphic inside L and, as is not difficult to verify, on Γ admits the estimate

$$|\mu(z)|\le\exp\{-C|\operatorname{cosec}\varphi|^{2l}\}. \tag{9}$$

It is easy to see that the function $f(Kz)\mu(z)$ $(K>0)$, holomorphic inside L, is continuous in the closure of L and, consequently, by a well-known theorem, is representable by the Cauchy integral over Γ. Therefore the equality

$$f(z)=\frac{1}{2\pi i\mu(zK^{-1})}\int_{\Gamma}\frac{f(K\zeta)\mu(\zeta)}{\zeta-zK^{-1}}\,d\zeta,\qquad (zK^{-1}\in L).$$

Putting here $K=r$, observing that L contains the unit disk inside it, and using (1) and (9), we obtain the estimate

$$|f(z)|\le C\max_{\zeta\in\Gamma}|f(r\zeta)\mu(\zeta)|\le C\max_{0\le\vartheta<2\pi}\left\{\exp\left(\frac{Cr^{\alpha}}{|\sin\vartheta|^{l}}-\frac{C}{|\sin\vartheta|^{2l}}\right)\right\}.$$

By the usual methods of finding the maximum we arrive at the estimate

$$|f(z)|\le Ce^{Cr^{2\alpha}}.$$

In order to finish the proof of the theorem, it remains, taking (1) into account, to apply the Phragmén–Lindelöf principle in angles of opening $<\pi(2\alpha)^{-1}$ containing the positive and the negative semiaxes of the real axis.

Remark 1. By modifying our arguments slightly, one can refine the theorem proved, replacing $\alpha=\text{const}$ by the refined order of Valiron.

Remark 2. It follows from the theorem proved that for $\alpha\le 1$ the function $f(z)$ is a function of growth not higher than first order. If the dependence of the constants in (1) on α and β is taken into account, then a more precise estimate of growth can be obtained.

Corollary. If an entire function $f(z)$ admits the lower estimate

$$|f(z)|\ge\exp\{-Cr^{\alpha}(|y|^{-k}+1)\}\qquad(0\le\alpha<1,\ k\ge 0), \tag{10}$$

then it is an entire function of finite degree, and for it the integral

$$\int_{-\infty}^{\infty} \frac{\ln_+ |f(x)|}{1+x^2}\,dx \tag{11}$$

converges.

Indeed, by the theorem proved, the order of the function $f(z)$ is finite in any case. Condition (10) ensures the possibility of representing $\ln[|f(z)|^{-1}]$ ([(2), p. 311]) and, consequently, also $\ln|f(z)|$, for $y > \delta$ ($\delta > 0$) in the form

$$\ln|f(z)| = \frac{y-\delta}{\pi}\int_{-\infty}^{\infty} \frac{\ln|f(t+i\delta)|}{(x-t)^2+(y-\delta)^2}\,dt + C(y-\delta). \tag{12}$$

and, moreover, from this equation it follows (([2]), p. 315) that

$$\int_{-\infty}^{\infty} \frac{\ln_+ |f(x+i\delta)|}{1+x^2}\,dx < \infty.$$

From representation (12) it follows (([2]), p. 301) that $f(z)$ is of finite order in the angle $|\arg z - \pi/2| \leqslant \pi/2 - \varepsilon$, whatever $\varepsilon > 0$ may be. In an analogous way we obtain that $f(z)$ is of finite order in the angle $|\arg z + \pi/2| \leqslant \pi/2 - \varepsilon$. Applying the Phragmén-Lindelöf principle in the angles $|\arg z| < 2\varepsilon$ and $|\pi - \arg z| < 2\varepsilon$, we arrive at the conclusion that $f(z)$ is an entire function of finite order. Now one may assert (([2]), p. 311) that the function $\ln|f(z)|$ is representable for $y \leqslant \delta$ ($\delta > 0$) in the form (a_k are the zeros of $f(z)$)

$$\ln|f(z)| = \frac{|y-\delta|}{\pi}\int_{-\infty}^{\infty} \frac{\ln|f(x+i\delta)|}{(x-t)^2-(y-\delta)^2}\,dt + C|y-\delta| + \sum_k \ln\left|\frac{z-a_k}{z-(a_k+2i\delta)}\right|,$$

whence we easily find that the integral (11) converges.

Let us note that the corollary proved is a generalization of the following theorem of M. G. Krein ([3]).

If $f(z)$ is an entire function with real zeros, such that the representation

$$\frac{1}{f(z)} = \sum_{k=1}^{\infty} \frac{A_k}{z-h_k} \quad \left(\sum_{k=1}^{\infty} |A_k| < \infty\right), \tag{13}$$

holds, then $f(z)$ is a function of finite order with convergent integral (11).

Indeed, from representation (13) it follows directly that

$$f(z) \geqslant \left(\sum_{k=1}^{\infty} |A_k|\right)^{-1} |y|,$$

and this lower estimate is more precise than (10).

Let us formulate one more result, obtained by the same method as our principal theorem.

If an entire function $f(z)$ admits the lower estimate

$$|f(z)| \geqslant \exp\left\{-T\left(\frac{r}{|\sin\varphi|^k}\right)\right\},$$

where $T(t)$ is a nondecreasing nonnegative function, $T^{(n)}(0) = 0$ $(n = 0, 1, \dots, p)$, $p \geqslant 1$, $k \geqslant 0$, $f(0) = 1$, and if

$$\int_0^\infty \frac{T(t)}{t^{1+\alpha(t)}}\,dt < \infty,$$

where $\alpha(t)$ is a function satisfying the conditions

$$p < \underline{\lim}_{t\to 0}(\alpha(t) + \alpha'(t)t\ln t) \leqslant \overline{\lim}_{t\to 0}(\alpha(t) + \alpha'(t)t\ln t) < p+1,$$

$$p < \underline{\lim}_{t\to \infty}(\alpha(t) + \alpha'(t)t\ln t) \leqslant \overline{\lim}_{t\to \infty}(\alpha(t) + \alpha'(t)t\ln t) < p+1,$$

then

$$\sum_m \frac{1}{|a_m|^{\alpha(|a_m|)}} \leqslant C(\alpha, k)\int_0^\infty \frac{T(t)}{t^{1+\alpha(t)}}\,dt,$$

where a_m are the zeros of the function $f(z)$.

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Note: Figure translations are in progress. See original paper for figures.