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Abstract

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THE RELATIVISTIC KINETIC EQUATION AND THE EQUILIBRIUM STATE OF A GAS IN A STATIC SPHERICALLY SYMMETRIC GRAVITATIONAL FIELD

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In this paper a kinetic equation is derived for a particle located in a static spherically symmetric gravitational field. The derivation of the equation is based on the results of papers ($^{1-3}$). This derivation is also valid in the case when the rest mass of the particle is zero. Next, the solution of the kinetic equation corresponding to the equilibrium state of a gas in this field is found.

In order to include particles with zero rest mass in the general scheme of consideration, a definition of proper time suitable also for such particles is necessary. As such a definition we choose the following: the interval of proper time $d\tau$ of a particle is the proportionality coefficient between the displacement vector dx along its world trajectory and its four-dimensional momentum: $dx = P d\tau$. It follows from this that

$$g_{ik} \frac{dx_i}{d\tau} \frac{dx_k}{d\tau} = m^2 c^2, \tag{1}$$

where g_{ik} is the metric tensor in event space; dx_i are the components of the displacement vector along the particle' s world trajectory; m is the rest mass of the particle; c is the speed of light.

The metric form of event space in the static spherically symmetric case has the form

$$ds^2 = V^2(\rho) c^2 dt^2 - G^2(\rho) d\rho^2 - \rho^2 (d\theta^2 + \sin^2\theta \, d\varphi^2). \tag{2}$$

The equations of motion of a particle in this case can be written as follows:

$$\frac{d\rho}{d\tau} = \frac{p_1}{G(\rho)}, \quad \frac{d\theta}{d\tau} = \frac{p_2}{\rho}, \quad \frac{d\varphi}{d\tau} = \frac{p_3}{\rho \sin\theta}, \quad \frac{dt}{d\tau} = \frac{p_4}{V(\rho)},$$

$$\frac{dp_1}{d\tau} = \frac{p_2^2 + p_3^2}{\rho G(\rho)} - \frac{c^2 p_4^2}{V(\rho) G(\rho)} \frac{dV(\rho)}{d\rho}, \tag{3}$$

$$\frac{dp_2}{d\tau} = -\frac{p_1 p_2}{\rho G(\rho)} + \frac{p_3^2 \cos\theta}{\rho \sin\theta}, \quad \frac{dp_3}{d\tau} = -\frac{p_1 p_3}{\rho G(\rho)} - \frac{p_2 p_3 \cos\theta}{\rho \sin\theta}.$$

The quantities p_1, p_2, p_3 are the spatial components of the particle momentum at the point ρ, θ, φ, t, and p_4 is its mass of motion. By virtue of (1) and (2),

$$p_4 = \sqrt{m^2 + p^2/c^2}, \qquad p^2 = p_1^2 + p_2^2 + p_3^2.$$

The right-hand sides of equations (3) are the components of the vector field $f(M)$ in the space F of states of the particle. By the state of a particle we mean the totality of its space-time position and momentum. As coordinates $x_1, \dots, x_7$ of the point M of the particle-state space we choose the space-time coordinates $x_1 = \rho,\ x_2 = \theta,\ x_3 = \varphi$;

$x_4 = t$ of the particle and the three spatial components $x_5 = p_1,\ x_6 = p_2,\ x_7 = p_3$ of its momentum. It is convenient to take as the volume element in F the 7-linear skew-symmetric form $\varepsilon(M; d_1, \dots, d_7)$, which on vectors of elementary displacements along the coordinate lines takes the value

$$\varepsilon(M; d_1, \dots, d_7) = \rho^2 G(\rho) V(\rho) \sin\theta \, d\rho \, d\theta \, d\varphi \, dt \, \frac{dp_1 dp_2 dp_3}{p_4}. \tag{4}$$

As the area element $d\Sigma$ on any hypersurface S in the space of states of the particle it is convenient to choose the 6-linear skew-symmetric form $\varepsilon(M; f(M), d_1, \dots, d_6)$, where $d_1, \dots, d_6$ are vectors of elementary displacements from the point M along the hypersurface S. In expanded notation the element $d\Sigma$ is expressed as follows:

$$d\Sigma = \frac{\rho^2 G(\rho) V(\rho) \sin\theta}{p_4} \operatorname{mod} \begin{vmatrix} f_1 & f_2 & \dots & f_7 \\ d_1 x_1 & d_1 x_2 & \dots & d_1 x_7 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_6 x_1 & d_6 x_2 & \dots & d_6 x_7 \end{vmatrix}, \tag{5}$$

where $f_i = dx_i/dt$ are defined by formulas (3), and mod denotes the modulus.

It is easy to verify that the exterior derivative of the 6-linear skew-symmetric form $\varepsilon(M; f(M), d_1, \dots, d_6)$ is zero, i.e.

$$\sum_{i=1}^{7} \frac{\partial}{\partial x_i} \left[\frac{\rho^2 G(\rho) V(\rho) \sin\theta}{p_4} f_i \right] = 0. \tag{6}$$

On the hypersurface $t = \text{const}$ the element $d\Sigma$ is equal to

$$d\Sigma = G(\rho)\rho^2 \sin\theta \, d\rho \, d\theta \, d\varphi \, dp_1 dp_2 dp_3. \tag{7}$$

Consequently, if the distribution density $A(M)$ of the probabilities of the particle' s position and of the components of its momentum is known for every value of t, then the probability of intersection of the particle' s world line with any hypersurface S is also known and is equal to

$$w(S) = \int_S \cdots \int A(M) \, \varepsilon(M; f(M), d_1, \dots, d_6). \tag{8}$$

Let the following quantities be given: a) for each region $D \subset F$, the probability

$$\iint_D \cdots \int \Delta(M) \, \varepsilon(M; d_1, \dots, d_7)$$

of formation of the particle in a state from this region; b) the probability $T^{-1}(M)$ of decay of the particle per unit of its proper time; c) for each region $D \subset F$, the probability

$$\iint_D \cdots \int K(M; M') \, \varepsilon(M; d_1, \dots, d_7)$$

of restoration of the particle in a state from the region D, under the condition that it has decayed in the state M'. If, in the process of decay and restoration of the particle, the quantities $T^{-1}(M)$ and $K(M; M')$ do not change, then, as shown in (1), $A(M)$ obeys the following kinetic equation:

$$\sum_{i=1}^{7} f_i \frac{\partial A(M)}{\partial x_i} + \frac{A(M)}{T(M)} = \lambda \iint_F \cdots \int \frac{A(M')K(M; M')}{T(M')} \, \varepsilon(M'; d_1', \dots, d_7') + \Delta(M). \tag{9}$$

In deriving this equation we have taken into account equality (6). The meaning of the parameter λ is the same as in (1).

The beginning of a collision of the particle under consideration with some particle of matter is its decay in the sense adopted by us, since

during a collision the motion of a particle is not specified by equations (3). Likewise, the termination of a collision is the restoration of the particle. The quantities $T^{-1}(M)$ and $K(M; M')$, which characterize these phenomena, depend on the distribution of the particles of matter with respect to coordinates and momenta. This dependence was found in (^{3}) under the condition that gravitational phenomena may be neglected. But here, in view of the smallness of the interaction cross sections of the particles, we are dealing precisely with such a case. Using the results of (^{3}), we can write the kinetic equation for the motion of a particle in matter in the presence of a gravitational field. Let us endow the particle under consideration and all its characteristics with the index α, and an arbitrary particle of matter and, correspondingly, its characteristics with the index β. Introduce the following notation: x is the set of coordinates ρ, θ, φ, t; $\mathbf{p}$ is the set of spatial components p_1, p_2, p_3 of the momentum of particle α; $\mathbf{q}$ is the set of spatial components q_1, q_2, q_3 of the momentum of particle β;

$$P = (p_1, p_2, p_3, p_4), \qquad Q = (q_1, q_2, q_3, q_4),$$

$$p_4 = \sqrt{m_\alpha^2 + p^2/c^2}, \qquad q_4 = \sqrt{m_\beta^2 + q^2/c^2};$$

$$\langle P, Q\rangle = c\sqrt{\left(p_4 q_4 - \frac{\mathbf{pq}}{c^2}\right)^2 - m_\alpha^2 m_\beta^2}, \qquad \mathbf{pq} = p_1 q_1 + p_2 q_2 + p_3 q_3;$$

$\sigma_{\alpha\beta}(\langle P, Q\rangle)$ is the interaction cross section of particles α and β, whose momenta are P and Q;

$$d\sigma = H_{\alpha\beta}(\langle P, Q\rangle, \langle P', Q\rangle, \langle P', P\rangle)\frac{dp_1' dp_2' dp_3'}{p_4'}$$

is the differential scattering cross section of particle α with momentum P by particle β with momentum Q; T_0^{-1} is the probability of spontaneous decay of particle α per unit of its proper time. The kinetic equation for the motion of a particle in matter in the presence of a gravitational field has the form

$$\sum_{i=1}^{7} f_i \frac{\partial A_\alpha(M)}{\partial x_i} + \frac{A_\alpha(M)}{T_\alpha(M)} = \lambda \int_{-\infty}^{\infty}\!\!\iint A_\alpha(x, \mathbf{p}') L_\alpha(M; \mathbf{p}')\frac{dp_1' dp_2' dp_3'}{p_4'} + \Delta(M), \tag{10}$$

where

$$T_\alpha^{-1}(M) = T_0^{-1} + \sum_\beta T_{\alpha\beta}^{-1}(M), \qquad L_\alpha(M; \mathbf{p}') = \sum_\beta L_{\alpha\beta}(M; \mathbf{p}'), \tag{11}$$

$$T_{\alpha\beta}^{-1}(M) = \int_{-\infty}^{\infty}\!\!\iint A_\beta(x, \mathbf{q})\,\langle P, Q\rangle\,\sigma_{\alpha\beta}(\langle P, Q\rangle)\frac{dq_1 dq_2 dq_3}{q_4}, \tag{12}$$

$$L_{\alpha\beta}(M; \mathbf{p}') = \int_{-\infty}^{\infty}\!\!\iint A_\beta(x, \mathbf{q}')\,\langle P', Q'\rangle\,H_{\alpha\beta}(\langle P', Q'\rangle, \langle P, Q'\rangle, \langle P, P'\rangle)\times$$

$$\times \frac{dq'_1 dq'_2 dq'_3}{q'_4}. \tag{13}$$

The summation in (11) is carried out over all particles of matter (particles with zero rest mass may also be included in the composition of matter).

We now turn to finding the distribution function corresponding to the equilibrium state of the gas in the field under consideration. We shall seek the distribution function $A_\alpha(M)$ in the form

$$A_\alpha(M) = a_\alpha(\rho)\exp\{-b(\rho)p_4c^2\}. \tag{14}$$

Here the index α assumes as many values as there are particles in the gas. As shown in paper [4], the collision integral for function (14) is equal to zero. The kinetic equation takes the form

$$\sum_{i=1}^{7} f_i \frac{\partial A_\alpha(M)}{\partial x_i} = 0. \tag{15}$$

It follows from this that (14) is equal to

$$A_\alpha(M) = a_\alpha \exp\left\{-\frac{V(\rho)\,p_\alpha c^2}{kT}\right\}, \tag{16}$$

where a_α = const; k is Boltzmann' s constant; T is a constant having the meaning of the temperature of the gas.

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Note: Figure translations are in progress. See original paper for figures.

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