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Soviet-era science, translated into English

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1960

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**Abstract**

**Full Text**

**K. I. BABENKO**

**ON THE APPROXIMATION OF PERIODIC FUNCTIONS OF MANY VARIABLES BY TRIGONOMETRIC POLYNOMIALS**

*(Presented by Academician M. V. Keldysh on 31 XII 1959)*

§ 1. Let us consider some questions in the theory of approximation of periodic functions of many variables. Whereas the corresponding theory for functions of one variable has been exhaustively developed, for functions of many variables the basic questions of the theory have remained open. This is explained, in our opinion, by the fact that in the theory of approximation of functions of many variables there have been no proper formulations of the questions.

In order to do this, let us consider the simplest problem of approximation of periodic functions in  $\mathcal{L}^2$ . Let  $E_m$  be an affine  $m$ -dimensional space,  $x = (x_1, x_2, \dots, x_m)$  a point of  $E_m$ , and let  $K$  be the unit cube in  $E_m$ :  $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \dots, 0 \leq x_m \leq 1$ . We shall consider functions  $f(x) = f(x_1, x_2, \dots, x_m)$ , periodic with period 1 in each variable. Denote by  $\mathcal{H}$  the Hilbert space of functions  $f(x)$  square-integrable over the cube  $K$ .

Usually in approximation theory the question is posed as follows: a class  $F$  of functions  $f$  and a system of functions  $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$  are given. It is required to estimate or, in the best case, determine

$$\rho_n = \sup_{f \in F} \inf_{c_k} \left\| f - \sum_{k=1}^n c_k \varphi_k \right\|.$$

When we consider functions of many variables, we cannot retain this formulation of the question; more precisely, we cannot specify a system of functions  $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ , or, in other words, we cannot in advance order according to some rule the system of functions  $\exp[2\pi i(k_1 x_1 + \dots + k_m x_m)]$ ,  $k_1 = 0, \pm 1, \dots; \dots; k_m = 0, \pm 1, \dots$ . In the present case we must only fix the class  $F$  and then choose such a method of ordering the system  $\{\exp[2\pi i(k_1 x_1 + \dots + k_m x_m)]\}$  that the quantity  $\rho_n$  be as small as possible.

In 1936 there appeared a remarkable paper by A. N. Kolmogorov <sup>(1)</sup>, in which he poses the question of finding such functions  $\varphi_1, \varphi_2, \dots, \varphi_n$  for which  $\rho_n$  is minimal under the condition that the class  $F$  is given. In the same paper A. N. Kolmogorov, for two classes of functions of one variable, determined the extremal systems of functions  $\varphi_1, \varphi_2, \dots, \varphi_n$  and the extremal values  $\rho_n$ . Thus, if  $F$  is the class of functions differentiable  $p$  times, subject to the condition

$$\int_0^1 |f^{(p)}(x)|^2 dx \leq 1,$$

then the corresponding quantity  $\rho_n(F)$  will be

$$\rho_n(F) = \left(\frac{1}{\pi n}\right)^p + O\left(\frac{1}{n^{p+1}}\right).$$

That is, the quantity  $\rho_n(F)$  has the same order as in the case of approximation of functions of the class by polynomials. Therefore, from the applied point of view it is quite immaterial whether one approximates functions of our class by polynomials or by “extremal” functions for the given  $F$ . In the case of functions of many variables the situation is exactly the opposite, as we shall see somewhat below.

Let us introduce the classes of functions under consideration. Put

$$p_1 = \frac{1}{i} \frac{\partial}{\partial x_1}, \quad p_2 = \frac{1}{i} \frac{\partial}{\partial x_2}, \dots, \quad p_m = \frac{1}{i} \frac{\partial}{\partial x_m},$$

and let the operators  $P_j(D) = P_j(p_1, p_2, \dots, p_m)$  be polynomials in  $p_1, p_2, \dots, p_m$  with constant coefficients. Consider the class  $F_p$  of functions  $f$  subject to the condition

$$\sum_j \int_K |P_j(D)f|^2 dx_1 \dots dx_m = \sum_j \int_K |P_j(D)f|^2 dx \leq 1. \quad (1)$$

For simplicity we shall restrict ourselves to the case of polynomials for which the differential operator  $\sum_j P_j^*(D)P_j(D) = Q(D)$  may have characteristic manifolds lying only in coordinate hyperplanes. The latter restriction is not essential and is made only for simplicity of exposition.

Consider the linear aggregate of solutions of the equation

$$Q(D) = \sum_j P_j^*(D)P_j(D) = 0$$

with boundary conditions of periodicity in the cube  $K$ . Closing this linear aggregate in  $\mathcal{H}$ , we obtain a subspace  $\mathcal{L}$ . By virtue of our condition on the characteristics of the operator  $Q(D)$ ,  $\mathcal{L}$  consists of those and only those functions which can be represented as the sum of no more than  $m$  functions of a smaller number of variables. Let  $\mathcal{H}/\mathcal{L}$  be the quotient space of  $\mathcal{H}$  by  $\mathcal{L}$ . Then in  $\mathcal{H}/\mathcal{L}$  the operator  $Q(D)$  has a complete system of eigenvectors. Denote  $n = (n_1, n_2, \dots, n_m)$ , where  $n_1, n_2, \dots, n_m$  are integers, and let  $nx = n_1x_1 + n_2x_2 + \dots + n_mx_m$ .

It is quite clear that the eigenvectors of the operator  $Q(D)$  will be the functions equivalent to  $e^{2\pi i n x}$ , with corresponding eigenvalues

$$\lambda(n) = \sum_j P_j^*(2\pi n_1, \dots, 2\pi n_m) P_j(2\pi n_1, \dots, 2\pi n_m), \quad \lambda > 0.$$

Number, as usual, the eigenvalues in increasing order

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots,$$

and let  $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$  be the corresponding eigenvectors. In what follows we shall always take as a representative of a class of the space  $\mathcal{H}/\mathcal{L}$  a function orthogonal to  $\mathcal{L}$ . Accordingly, the scalar product in  $\mathcal{H}/\mathcal{L}$  is defined as usual: if  $f$  and  $g$  are representatives of the classes  $\xi$  and  $\eta$ , then we put  $(\xi, \eta) = (f, g)$ .

Let the vector  $f$  of the space  $\mathcal{H}/\mathcal{L}$  belong to the aggregate  $F_p$ . Consider the problem of approximation of this vector by  $n$  elements from  $\mathcal{H}/\mathcal{L}$ . We may assume these elements  $f_1, f_2, \dots, f_n$  to be orthonormal. Let

$$\rho_n^2(f) = \|f\|^2 - \sum_{k=1}^n |(f, f_k)|^2.$$

Consider

$$\sup_{f \in F_p} \rho_n(f) = \rho_n(F_p; f_1, f_2, \dots, f_n),$$

and let

$$\rho_n(F_p) = \inf_{f_1, \dots, f_n} \rho_n(F_p; f_1, f_2, \dots, f_n).$$

It is easy to show that  $\rho_n(F_p) = 1/\sqrt{\lambda_{n+1}}$ . Indeed, if we take as  $f_1, f_2, \dots, f_n$  the eigenvectors  $\varphi_1, \varphi_2, \dots, \varphi_n$ , then we obtain

$$\rho_n(F_p; \varphi_1, \varphi_2, \dots, \varphi_n) = 1/\sqrt{\lambda_{n+1}}.$$

Indeed, let  $f \in F_p$ . By the completeness of the system of eigenfunctions of the operator  $Q(D)$ ,

$$f = \sum_1^{\infty} \alpha_k \varphi_k.$$

Condition (1) is written as follows:

$$\sum_1^{\infty} \lambda_k |\alpha_k|^2 \leq 1. \quad (2)$$

Since

$$\rho_n^2(f) = \sum_{n+1}^{\infty} |\alpha_k|^2,$$

it is evident that

$$\rho_n(F_p; \varphi_1, \dots, \varphi_n) \leq 1/\sqrt{\lambda_{n+1}}.$$

Suppose that  $\rho_n(F_p) < 1/\sqrt{\lambda_{n+1}}$  and that  $g_1, g_2, \dots, g_n$  is an orthonormal system of vectors for which

$$\rho_n(F_p; g_1, g_2, \dots, g_n) < 1/\sqrt{\lambda_{n+1}}.$$

Put

$$g_k = \sum_{\nu=1}^{\infty} \beta_{k\nu} \varphi_{\nu}, \quad k = 1, 2, \dots, n.$$

It is clear that

$$\det(\beta_{k\nu})_{k,\nu=1}^n \neq 0.$$

We shall show that  $\beta_{k,n+1} = 0$ ,  $k = 1, 2, \dots, n$ . Assuming the contrary, we find a system of numbers  $\alpha_{\nu}$ ,  $\nu = 1, 2, \dots, n+1$ , for which the relations

$$\sum_{\nu=1}^{n+1} \bar{\beta}_{k\nu} \alpha_{\nu} = 0, \quad \sum_{\nu=1}^{n+1} \lambda_{\nu} |\alpha_{\nu}|^2 = 1, \quad k = 1, 2, \dots, n \quad (3)$$

hold. Taking the function

$$f(x) = \sum_1^{n+1} \alpha_{\nu} \varphi_{\nu},$$

we obtain

$$\rho_n^2(f) = \|f\|^2 - \sum_1^n |(f, g_k)|^2 = \|f\|^2 = \sum_1^{n+1} |\alpha_{\nu}|^2 < \frac{1}{\lambda_{n+1}},$$

which contradicts condition (3). Thus,  $\beta_{k,n+1} = 0$ ,  $k = 1, 2, \dots, n+1$ . Now take

$$f = 1/\sqrt{\lambda_{n+1}} \varphi_{n+1}.$$

Then

$$\rho_n^2(f) = \|f\|^2 < \frac{1}{\lambda_{n+1}},$$

which is absurd. Therefore,

$$\rho_n(F_p) = \frac{1}{\sqrt{\lambda_{n+1}}}. \quad (4)$$

Consequently, for the class  $F_p$  the extremal system of functions will be the system of eigenvectors of the operator  $Q(D)$ . We leave open the question of uniqueness, up to a unitary transformation, of the extremal system of functions.

§ 2. Let us consider the more complicated question of uniform approximation of functions in the cube  $K$ . In this case, to indicate the best systems of functions seems to us very difficult. Nevertheless, the results of § 1 allow—

make it possible in many cases to indicate such systems of functions that will be asymptotically exact as  $n \rightarrow \infty$ .

Let  $F_P$  be the class of functions  $f$  subject to the condition

$$\sup_{x \in K} \left( \sum_j |P_j(D)f|^2 \right)^{1/2} \leq 1. \quad (5)$$

Denote, as usual, by  $C = C_K$  the space of functions continuous and periodic on the cube  $K$ , with norm  $\|f\| = \max_K |f(x)|$ .

Consider the totality of solutions of the equation

$$\sum_j |P_j(D)f|^2 = 0.$$

This is a linear manifold in  $C$ . Closing it in  $C$ , we obtain a linear subspace  $\mathcal{L}$  in  $C$ . In the quotient space  $C/\mathcal{L}$  define the norm in the usual way. Let  $f \in F_P$ , and denote also by  $f$  the corresponding class from  $C/\mathcal{L}$ . Put, as above,

$$\rho_n(F_P; f_1, f_2, \dots, f_n) = \sup_{f \in F_P} \inf_{c_k} \left\| f - \sum_1^n c_k f_k \right\|,$$

where  $f_1, f_2, \dots, f_n \in C/\mathcal{L}$ , and let

$$\rho_n(F_P) = \inf_{f_1, f_2, \dots, f_n} \rho_n(F_P; f_1, f_2, \dots, f_n).$$

Assuming that the characteristic manifolds of the operator  $Q(D)$  satisfy the condition posed earlier, one may assert that

$$\frac{A}{\sqrt{\lambda_{n+1}}} < \rho_n(F_P) < \frac{B}{\sqrt{\lambda_{n+1}}}, \quad (6)$$

where the constants  $A$  and  $B$  do not depend on  $n$ . The eigenvectors  $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$  also in this case give the “best” system of functions, since

$$\rho_n(F_P; \varphi_1, \varphi_2, \dots, \varphi_n) \leq \frac{C}{\sqrt{\lambda_{n+1}}}. \quad (7)$$

We cannot prove the inequality

$$\rho_n(F_P) > \frac{A}{\sqrt{\lambda_{n+1}}}$$

in full generality.

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Received  
31 XII 1959

## REFERENCES

1. A. Kolmogoroff, *Ann. of Math.*, **37**, No. 1 (1936).

*Note: Figure translations are in progress. See original paper for figures.*

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