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ON LOCAL THEOREMS FOR LARGE DEVIATIONS

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Abstract

Full Text

MATHEMATICS

V. V. PETROV

ON LOCAL THEOREMS FOR LARGE DEVIATIONS

(Presented by Academician V. I. Smirnov on V 16, 1960)

1. Consider a sequence of independent random variables $X_1, X_2, ...$, having the same distribution with finite variance $\sigma^2 > 0$ and mathematical expectation EX_1 equal to zero. Put

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n}{\sigma\sqrt{n}}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt, \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The distribution function of the random variable Z_n will be denoted by $F_n(x)$, and the corresponding density of the distribution, if it exists, by $p_n(x)$. Studying the behavior of $F_n(x)$ as $x \to \infty$ together with n, Cramér [1] obtained a fundamental theorem, which was later generalized by the author [2] to the case of non-identically distributed variables, with a concomitant strengthening of it for the special case of identical distributions. We give the statement of Cramér' s theorem with this strengthening taken into account.

If, for some $a > 0$,

$$Ee^{a|X_1|} < \infty, \tag{A}$$

then for $x > 1$, $x = o(\sqrt{n})$, and $n \to \infty$ we have

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\left[\frac{x^3}{\sqrt{n}}\lambda\left(\frac{x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\left[-\frac{x^3}{\sqrt{n}}\lambda\left(-\frac{x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where

$$\lambda(t) = c_0 + c_1 t + c_2 t^2 + \cdots \tag{1}$$

is a power series converging for all sufficiently small values of $|t|$.

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Denote by γ_m the cumulant of order m of the random variable X_1. The coefficient c_k in Cramér' s series (1) is expressed only in terms of $\gamma_2 = \sigma^2, \gamma_3, \dots, \gamma_{k+3}$. In particular, $c_0 = \gamma_3/(6\sigma^3)$, $c_1 = (\sigma^2\gamma_4 - 3\gamma_3^2)/(24\sigma^6)$.

Analogous results, under the assumption that condition (A) is satisfied, were obtained by V. Richter [3] for local theorems.

Yu. V. Linnik [4,5] obtained a number of integral and local limit theorems for the case when Cramér' s condition (A) is not satisfied and therefore the previously used methods are inapplicable. In [4,5] the problem of normal convergence was studied, i.e., the problem of determining conditions under which, for example, $[1 - F_n(x)]/[1 - \Phi(x)] \to 1$ or $p_n(x)/\varphi(x) \to 1$ as $n \to \infty$ and $0 \leq x \leq \psi(n)$, where $\psi(n)$ is a monotone function, $\lim_{n\to\infty} \psi(n) = +\infty$.

However, the method proposed by Yu. V. Linnik can also be applied to the solution of other problems. We formulate several results obtained in this way.

If s is a nonnegative integer, then by $\lambda^{[s]}(t)$ we shall denote the segment of Cramér' s series (1) consisting of its first s terms. Thus,

$$\lambda^{[s]}(t) = \sum_{k=0}^{s-1} c_k t^k \quad (s \geqslant 1);$$

in the case $s = 0$ we put $\lambda^{[0]}(t) \equiv 0$.

Theorem 1. *Suppose that for some n_0 there exists a bounded density $p_n(x)$ of the distribution of the random variable Z_n. If the condition*

$$\mathbf{E} \exp |X_1|^{\frac{4\alpha}{2\alpha+1}} < \infty \tag{2}$$

is satisfied for some α, $0 < \alpha < 1/2$, then, for $|x| \leqslant n^\alpha/\rho(n)$, where $\rho(n)$ is any monotone function such that $\lim\limits_{n\to\infty} \rho(n) = +\infty$, as $n \to \infty$ we have, uniformly with respect to x,

$$\frac{p_n(x)}{\varphi(x)} = \exp\left[\frac{x^3}{\sqrt{n}} \lambda^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right][1 + o(1)], \tag{3}$$

where s is a nonnegative integer determined by the inequalities

$$\frac{s}{2(s+2)} < \alpha \leqslant \frac{s+1}{2(s+3)}. \tag{4}$$

The following theorem shows that condition (2) is necessary in order that (3) hold as $n \to \infty$ uniformly with respect to x in the interval $|x| \leqslant n^\alpha \rho(n)$.

Theorem 2. *If (3) holds as $n \to \infty$, for $|x| \leqslant n^\alpha \rho(n)$ and some integer $s \geqslant 0$, uniformly with respect to x, where $\rho(n)$ is some monotone function and α is*

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some number such that $\lim_{n\to\infty} \rho(n) = +\infty$, $0 < \alpha < 1/2$, *then condition (2) is satisfied for the given* α.

As consequences of Theorems 1 and 2 we obtain certain results of Yu. V. Linnik. Suppose that for some n_0 there exists a bounded density $p_n(x)$. By $\rho(n)$ below is denoted a monotone function increasing (arbitrarily slowly) to infinity.

Corollary 1. Let $0 < \alpha \leqslant 1/6$. Condition (2) is sufficient in order that, for $|x| \leqslant n^\alpha/\rho(n)$ and $n \to \infty$, the relation

$$\frac{p_n(x)}{\varphi(x)} \to 1 \tag{5}$$

hold uniformly with respect to x, and is necessary in order that (5) hold for $|x| \leqslant n^\alpha \rho(n)$ and $n \to \infty$ uniformly with respect to x.

Corollary 2. Let α satisfy condition (4) with $s > 0$. Conditions (2) and

$$\gamma_m = 0 \quad (m = 3, \dots, s+2) \tag{6}$$

are sufficient in order that (5) hold for $|x| \leqslant n^\alpha/\rho(n)$ and $n \to \infty$ uniformly with respect to x, and are necessary in order that (5) hold for $|x| \leqslant n^\alpha \rho(n)$ and $n \to \infty$ uniformly with respect to x.

We now turn to the case where the random variables $X_1, X_2, \dots$ have a lattice distribution. Let $X_1, X_2, \dots$ be a sequence of identically distributed independent random variables, $\mathbf{E}X_1 = 0$, $0 < \mathbf{D}X_1 = \sigma^2 < \infty$. Suppose that X_1 has only possible values of the form $a + kh$ ($k = 0, \pm 1, \pm 2, \dots$), where a is some real number and h is the maximal span of the distribution. Introduce the notation

$$p_k = \mathbf{P}\{X_1 = a+kh\}, \quad P_n(N) = \mathbf{P}\left\{\sum_{j=1}^{n} X_j = an + Nh\right\}, \quad x = x_{nN} = \frac{an + Nh}{\sigma\sqrt{n}}.$$

Theorem 3. If condition (2) is satisfied for some α, $0 < \alpha < 1/2$, then for $|x| \leq n^\alpha/\rho(n)$, where $\rho(n)$ is any monotone function such that $\lim_{n\to\infty} \rho(n) = +\infty$, as $n \to \infty$ we have, uniformly with respect to x,

$$\frac{\frac{\sigma\sqrt{n}}{h} P_n(N)}{\varphi(x)} = \exp\left[\frac{x^3}{\sqrt{n}} \lambda^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right] [1 + o(1)]. \tag{7}$$

Here s is a nonnegative integer determined by means of (4).

Theorem 4. If (7) holds as $n \to \infty$ for some integer $s \geq 0$, uniformly with respect to x in the segment $|x| \leq n^\alpha \rho(n)$, where α is some number and $\rho(n)$ is

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some monotone function satisfying the conditions $0 < \alpha < 1/2$, $\lim_{n\to\infty} \rho(n) = +\infty$, then condition (2) is satisfied for the given α.

From Theorems 3 and 4 there follow corollaries analogous to the above-indicated corollaries from Theorems 1 and 2.

2. The results obtained can be generalized to the case of non-identically distributed random variables. We shall confine ourselves to the following assertion.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables,

$$\mathbf{E}X_j = 0, \qquad \mathbf{D}X_j = \sigma_j^2 < \infty \quad (j = 1, 2, \ldots).$$

Put $s_n^2 = \sum_{j=1}^n \sigma_j^2$,

$$Z_n = \frac{1}{s_n} \sum_{j=1}^n X_j.$$

By $V_j(x)$ denote the distribution function of the random variable X_j, and by $v_j(t)$ its characteristic function. As before, by $F_n(x)$ and $p_n(x)$ we shall denote respectively the distribution function and the density of the distribution of the random variable Z_n.

Theorem 5. Suppose that for some α, $0 < \alpha < 1/2$, the following conditions are satisfied:

I.

$$\mathbf{E}\exp\left\{|X_j|^{\frac{4\alpha}{2\alpha+1}}\right\} \le C \qquad (j = 1, 2, \ldots),$$

where C is a constant.

II.

$$\int_{|t|>\varepsilon} \prod_{j=1}^n |v_j(t)|\, dt = O\left(e^{-\delta n^{2\alpha}}\right)$$

for every $\varepsilon > 0$ and some $\delta > 0$.

III.

$$\lim_{n\to\infty} \frac{s_n^2}{n} > 0.$$

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Then, for all sufficiently large n, there exists a density $p_n(x)$, and for $|x| \leq n^\alpha/\rho(n)$, where $\rho(n)$ is an arbitrary monotone function such that $\lim_{n\to\infty} \rho(n) = +\infty$, we have

$$\frac{p_n(x)}{\varphi(x)} = \exp\left[\frac{x^3}{\sqrt{n}}\lambda_n^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right][1+o(1)]$$

as $n \to \infty$, uniformly with respect to x. Here s is a nonnegative integer determined by means of (4), and $\lambda_n^{[s]}(t)$ is the truncation of the series $\lambda_n(t)$ consisting of its first s terms. The series $\lambda_n(t)$ is defined in the work [(2)].

We note that the power series $\lambda_n(t)$, whose coefficients in the general case depend on n, in the particular case of identical distributions coincides with Cramér' s series $\lambda(t)$.

Theorem 1 follows from Theorem 5.

In conclusion, I express my deep gratitude to Yu. V. Linnik for his attention to the present work and for valuable advice.

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Note: Figure translations are in progress. See original paper for figures.

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