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Abstract

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SOME REMARKS ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENCE EQUATIONS

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In the present note we investigate the question of the asymptotic behavior of solutions of difference equations in the class of square-summable sequences. Similar questions were considered in the class of generalized functions by A. I. Zhukov (^{1}). However, there are examples in which the asymptotic solutions obtained by him do not belong to the class considered by us.

Consider the difference equation

$$u_k^{n+1} = \sum_p c_{p-k} u_p^n, \tag{1}$$

for which a difference analogue of the Cauchy problem holds.

Suppose that the initial data u_k^0 and the coefficients c_k of the difference scheme (1) belong to the class l_2. Introduce the notation

$$U^n(s) = \sum_{k=-\infty}^{+\infty} u_k^n \exp[iks]. \tag{2}$$

As is known (^{2}), the series (2) converges and the function $U^n(s)$ belongs to the class $L_2(-\pi, +\pi)$.

With the aid of (2), to equation (1) we associate the expression

$$U^n(s) = \lambda^n(s) U^0(s), \tag{3}$$

where $U^0(s)$ and $\lambda(s)$ belong to the class $L_2(-\pi, +\pi)$.

We shall call equation (1) stable (^{3}) if

$$\sum_k |u_k^n|^2 \leq M \sum_k |u_k^0|^2, \tag{4}$$

where $M > 0$ is a constant independent of n.

Inequality (4) is equivalent to the inequality

$$\int_{-\pi}^{+\pi} |U^n(s)|^2 \, ds \leq M \int_{-\pi}^{+\pi} |U^0(s)|^2 \, ds. \tag{5}$$

From (3) it is not difficult to conclude that, for (5) to hold for arbitrary initial data u_k^0, it is necessary and sufficient that the condition

$$|\lambda(s)| \leq 1. \tag{6}$$

The solution of (1) can be represented in the form

$$u_k^n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \lambda^n(s) U^0(s) \exp[-iks] \, ds. \tag{7}$$

Let $\lambda(s)$ satisfy the conditions:

$$\lambda(0) = 1, \qquad |\lambda(s)| < 1 \quad \text{for } s \neq 0 \tag{8}$$

and suppose that there exists some neighborhood of $s = 0$, $-\alpha \leq s \leq \alpha$, in which $\lambda(s)$ can be represented in the form

$$\lambda(s) = \exp\left[i \sum_{l=1}^{2p-1} \beta_l s^l - \beta_{2p} s^{2p} + \beta_{2p+1} s^{2p+1} v(s) \right], \tag{9}$$

where $\beta_{2p} > 0$, and $v(s)$ is a function analytic in the indicated neighborhood. Let

$${}^{(1)}u_k^n = \frac{1}{2\pi} \int_{-\alpha}^{+\alpha} \lambda^n(s) U^0(s) \exp[-iks] \, ds; \tag{10}$$

then from (8) it follows that

$$\sum_k |u_k^n - {}^{(1)}u_k^n|^2 \leq M_1 q^n, \tag{11}$$

where $q \leq q_0 < 1$, and M_1 is a constant independent of n.

It is not difficult to verify that the neighborhood of $s = 0$ can be so narrowed that the inequality

$$|\lambda(s)| \le \exp[-\beta s^{2p}], \tag{12}$$

where $\beta > 0$, will be satisfied.

Now let

$${}^{(2)}u_k^n = \frac{1}{2\pi}\int_{-\alpha}^{+\alpha} \exp\left[in\sum_{l=1}^{2p-1}\beta_l s^l - n\beta_{2p}s^{2p}\right] U^0(s)\exp[-iks]\,ds. \tag{13}$$

We shall try to determine the behavior of the expression

$$\sum_k |u_k^n - {}^{(2)}u_k^n|^2 \quad \text{as } n \to \infty.$$

Choose a sequence ε_n, whose rate of decrease will be specified later. Let

$$\bar{u}_k^n = \frac{1}{2\pi}\int_{-\varepsilon_n}^{+\varepsilon_n} \lambda^n(s)U^0(s)\exp[-iks]\,ds; \tag{14}$$

then, by virtue of (12),

$$\sum_k |{}^{(1)}u_k^n - \bar{u}_k^n|^2 \le M_2\exp[-2\beta n\varepsilon_n^{2p}]. \tag{15}$$

We now require that $n\varepsilon_n^{2p} = n^{\delta\cdot 2p}$, $\delta > 0$. Then $\varepsilon_n = n^{-1/2p+\delta}$ and $\delta < 1/2p$. With such a choice of δ we see that the error (15) decreases as n grows like an exponential function.

Now let

$$\bar{\bar{u}}_k^n = \frac{1}{2\pi}\int_{-\varepsilon_n}^{+\varepsilon_n} \exp\left[in\sum_{l=1}^{2p-1}\beta_l s^l - n\beta_{2p}s^{2p}\right] U^0(s)\exp[-iks]\,ds. \tag{16}$$

Let us estimate

$$\sum_k \left|\bar{u}_k^n - \bar{\bar{u}}_k^n\right|^2 = \frac{1}{2\pi}\int_{-\varepsilon_n}^{+\varepsilon_n} \left|\lambda^n(s) - \exp\left[in\sum_{l=1}^{2p-1}\beta_l s^l - n\beta_{2p}s^{2p}\right]\right|^2 |U^0(s)|^2\,ds. \tag{17}$$

We note that the square of the difference in (17) can be represented in the form

$$\left|\lambda^n(s) - \exp\left[in\sum_{l=1}^{2p-1}\beta_l s^l - n\beta_{2p}s^{2p}\right]\right|^2 =$$

$$= \beta_{2p+1}^2 n^2 s^2 \left|(2p+1)\xi^{2p}v(\xi) + \xi^{2p+1}v'(\xi)\right|^2 \times$$

$$\times \left|\exp\left[2n\beta_{2p+1}\xi^{2p+1}v(\xi)\right]\right| \exp\left[-2n\beta_{2p}s^{2p}\right], \qquad (18)$$

where ξ is contained between $-\varepsilon_n$ and $+\varepsilon_n$.

We require that $n\varepsilon_n^{2p+1} \to 0$. To this end the parameter δ should be chosen so that $\delta < 1/2p(2p+1)$. Hence we conclude that

$$\sum_k \left|\overline{u}_k^n - \overline{\overline{u}}_k^n\right|^2 \leq M_3 n^2 \int_{-\varepsilon_n}^{+\varepsilon_n} s^2 \xi^{4p} \exp\left[-2n\beta_{2p}s^{2p}\right] ds \leq M_4 n^{-3/2p+(4p+1)\delta}. \qquad (19)$$

Observing here that $\sum |u_k^n|^2$ and $\sum |\overline{u}_k^n|^2$ have order of growth $n^{-1/2p}$, we see that

$$\sum_k \left|u_k^n - \overset{(2)}{u}{}_k^n\right|^2 \leq M_5 n^{-3/2p+(4p+1)\delta}, \qquad (20)$$

i.e., the error tends to zero faster than the quantities under investigation themselves decrease.

Let us consider several examples of difference schemes and write down the asymptotic formulas for them.

Example 1.

$$u_k^{n+1} = \frac{u_{k+1}^n + u_{k-1}^n}{2} + \frac{\tau}{2h}\left(u_{k+1}^n - u_{k-1}^n\right). \qquad (21)$$

For this difference scheme, conditions (8) are satisfied if

$$r = \frac{\tau}{h} \leq 1.$$

The asymptotic solution has the form

$$\overset{(2)}{u}{}_k^n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \exp\left[inrs - n\frac{1-r^2}{2}s^2\right] U^0(s)\exp[-iks]\,ds. \qquad (22)$$

Example 2.

$$u_k^{n+1} = u_k^n + \frac{r}{2}\left(u_{k+1}^n - u_{k-1}^n\right) + \frac{r^2}{2}\left(u_{k+1}^n - 2u_k^n + u_{k-1}^n\right). \qquad (23)$$

Conditions (8) are the same as in the preceding example. The asymptotic solution of (23) has the form

$$\overset{(2)}{u}{}_k^n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \exp\left[inrs + in\frac{r}{6}(1-r)s^3 - n\frac{1}{8}r^2(1-r^2)s^4\right] U^0(s)\exp[-iks]\,ds. \qquad (24)$$

As follows from the article by A. I. Zhukov ([1]), in example 2 an asymptotic formula can be obtained in the sense of convergence in the class of generalized functions

$$\tilde{u}_k^n = \frac{1}{2\pi^3\sqrt[3]{n}}\int_{-\infty}^{+\infty} \cos\left[\frac{nr+k}{\sqrt[3]{n}}s + \frac{r}{6}(1-r)s^3\right] ds, \qquad U^0(s) = 1. \qquad (25)$$

However, it is not difficult to see that $\tilde{u}_k^n$ does not belong to the class l_2.

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Note: Figure translations are in progress. See original paper for figures.