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**Abstract**

**Full Text**

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## ON THE THEORY OF IDEALLY HARDENING MEDIA

*(Presented by Academician L. I. Sedov, 23 X 1959)*

In the work of V. Prager <sup>(1)</sup>, attention was drawn to a new branch of continuum mechanics: the theory of ideally hardening media. Prager's model, despite its extreme idealization, makes it possible to approach the study of a new class of mechanical phenomena and, serving as a basis for further constructions, deserves careful investigation.

**Fig. 1**

**Fig. 2**

In this note we consider certain questions in the theory of ideally hardening media. The medium under consideration will be assumed to be: 1) homogeneous, 2) isotropic, 3) incompressible. Further, we shall assume: 4) independence of the material behavior with respect to reversal of the sign of the stresses. Finally, suppose that: 5) the medium is ideally hardening. Let us explain the latter. Consider an element of the medium under the action of uniaxial tension-compression. We shall assume that the dependence  $\sigma-\varepsilon$  has the form shown in Fig. 1, i.e., the medium deforms freely until the deformation (in the general case, some combination of deformations) reaches a certain limiting value, which we denote by  $k$ .

Consider the space of principal deformations; the hardening condition is interpreted in it as a certain curve lying in the deviatoric plane  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ . In Fig. 2 the hexagons  $ABCDEF$  and  $A_1B_1C_1D_1E_1F_1$  are shown, between which lie all possible unbent curves of the hardening condition. In what follows we shall consider only unbent curves of the hardening condition, and the hardening condition will be regarded as a hardening potential <sup>(2,3)</sup>. Properly speaking, considerations concerning the appropriateness of such definitions can be carried over, with insignificant reservations, from the theory of ideal plasticity and are closely connected with considerations of uniqueness and extremality of the true processes <sup>(4, 5)</sup>.

Let us single out from all possible unbent hardening curves the unique one that determines the absolute extremality of the work for given boundary conditions. It is natural to expect that such a curve will prove to be either the hexagon  $ABCDEF$ , or the hexagon  $A_1B_1C_1D_1E_1F_1$ . In ana-

logical situation in the theory of ideal plasticity <sup>(6)</sup> the hexagon  $ABCDEF$  was taken. In the present case we shall likewise assume that the true processes differ

from all possible ones in that, for a prescribed deformation of an element of the body, they require the minimum work of deformation. Then in our case the true hardening condition is interpreted by the hexagon  $A_1B_1C_1D_1E_1F_1$ .

Let us write the equations of the sides of the hexagon  $A_1B_1C_1D_1E_1F_1$ . Consider, for example, the side  $F_1A_1$ . Its equation is written in the form

$$\varepsilon_1 - \frac{1}{2}\varepsilon_2 - \frac{1}{2}\varepsilon_3 = k, \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0. \quad (1)$$

The equation of the side  $A_1B_1$  is written in the form

$$-\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_1 = k, \quad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0. \quad (2)$$

In this case, evidently, the greatest freedom of the stressed state corresponds to the vertices of the hexagon  $A_1B_1C_1D_1E_1F_1$ ; for the vertex  $A_1$  we shall have

$$\varepsilon_1 = -\varepsilon_2 = \frac{2}{3}k, \quad \varepsilon_3 = 0. \quad (3)$$

In the case of a plane strain state, directing the  $z$ -axis along the axis of an infinitely long cylindrical body, suppose that the  $z$ -axis coincides with the third principal direction of strain. Then

$$\varepsilon_z = \varepsilon_3 = \varepsilon_{xz} = \varepsilon_{yz} = \tau_{xz} = \tau_{yz} = 0$$

and all components depend only on the coordinates  $x, y$ .

Considering conditions (1) and (2), it is easy to see that the case of plane strain state can correspond only to the vertices of the hexagon  $A_1B_1C_1D_1E_1F_1$ . In our case one must consider the vertex  $A_1$  or  $D_1$ . Condition (3) is written in the form

$$(\varepsilon_x - \varepsilon_y)^2 + 4\varepsilon_{xy}^2 = 4\chi^2 \quad \left( \chi^2 = \frac{4k^2}{9} \right). \quad (4)$$

We satisfy condition (4) by putting

$$\varepsilon_x = -\varepsilon_y = \chi \cos 2\theta, \quad \varepsilon_{xy} = \chi \sin 2\theta. \quad (5)$$

Substituting expressions (5) into the strain compatibility conditions

$$\frac{\partial q}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y} = 0,$$

$$\frac{\partial q}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y} = 0, \quad (6)$$

where

$$q = \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right), \quad \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$

we obtain a system of two equations of hyperbolic type, whose characteristics are written in the form

$$dy - \operatorname{tg} \theta dx = 0, \quad dy + \operatorname{ctg} \theta dx = 0. \quad (7)$$

Along the characteristics the relations

$$dq \mp \chi d\theta = 0, \quad (8)$$

hold.

Let us pass to the equations for the stresses. Differentiating relation (4), we find

$$ds_x = d(\sigma_x - \sigma) = 2d\mu(\varepsilon_x - \varepsilon_y) = 4d\mu\chi \cos 2\theta,$$

$$ds_y = d(\sigma_y - \sigma) = 2d\mu(\varepsilon_y - \varepsilon_x) = -4d\mu\chi \cos 2\theta, \quad (9)$$

$$2d\tau_{xy} = 8d\mu \varepsilon_{xy} = 8d\mu \sin 2\theta,$$

where  $\sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$ .

From (9) we obtain  $\sigma_z = \frac{1}{2}(\sigma_x + \sigma_y)$ . We write the equilibrium equations in the form

$$\frac{\partial \sigma}{\partial x} + \frac{\partial s_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial s_y}{\partial y} = 0. \quad (10)$$

Substituting the expressions (9) into (10), we obtain a system of two equations of hyperbolic type, whose characteristics coincide with the characteristics (7). Along the characteristics the relations hold

### Table 1

	$\vec{\xi}$	$\vec{\eta}$	$\vec{\zeta}$
$x$	$l_1$	$m_1$	$n_1$
$y$	$l_2$	$m_2$	$n_2$
$z$	$l_3$	$m_3$	$n_3$

$$d\sigma \mp \chi d\mu = 0. \quad (11)$$

Let us consider the spatial deformation of ideally hardening media. We shall restrict ourselves to consideration of the case of the condition of complete hardening (3), as providing the greatest freedom for the state of stress. Suppose that the directions of the principal strain rates  $\varepsilon_1, \varepsilon_2, \varepsilon_3$ , which we denote by  $\vec{\xi}, \vec{\eta}, \vec{\zeta}$ , form angles with the axes  $x, y, z$  whose cosines are given in Table 1. In what follows we shall assume that  $l_i = \cos \varphi_i$ ,  $m_i = \cos \psi_i$ ,  $n_i = \cos \theta_i$  ( $i = 1, 2, 3$ ). Using the transformation

$$\begin{aligned} \varepsilon_x &= \varepsilon_1 l_1^2 + \varepsilon_2 m_1^2 + \varepsilon_3 n_1^2, \\ \varepsilon_{xy} &= \varepsilon_1 l_1 l_2 + \varepsilon_2 m_1 m_2 + \varepsilon_3 n_1 n_2, \end{aligned} \quad \begin{pmatrix} x & y & z \\ 1 & 2 & 3 \end{pmatrix} \quad (12)$$

and taking expression (3) into account, we obtain

$$\begin{aligned} \varepsilon_x &= {}^2/3 k (\cos^2 \varphi_1 - \cos^2 \psi_1), \\ \varepsilon_{xy} &= {}^2/3 k (\cos \varphi_1 \cos \varphi_2 - \cos \psi_1 \cos \psi_2). \end{aligned} \quad \begin{pmatrix} x & y & z \\ 1 & 2 & 3 \end{pmatrix} \quad (13)$$

Here and in what follows the symbols  $(xyz)$ ,  $(123)$  mean that the unwritten relations are obtained by cyclic permutation of the indices. We write the compatibility conditions in the form

$$\begin{aligned} \frac{\partial \omega_z}{\partial x} - \frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial x} &= 0, \\ \frac{\partial \omega_z}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y} &= 0, \end{aligned} \quad (xyz) \quad (14)$$

where

$$\omega_x = \frac{1}{2} \left( \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right), \quad \omega_y = \frac{1}{2} \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right), \quad \omega_z = \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right).$$

Substituting the expressions (13) into the 6 relations (14), and adjoining the 3 relations valid between the cosines  $l_i, m_i$ , we obtain a system of 9 equations in the 9 unknowns:  $\varphi_i, \psi_i$  ( $i = 1, 2, 3$ ) and  $\omega_x, \omega_y, \omega_z$ . Denoting by  $\chi(x, y, z)$  the

equation of the characteristic surfaces of this system of equations, we find that the characteristic determinant reduces to the form

$$\chi_x \chi_y \chi_z (\text{grad } \chi \cdot \vec{\xi}) (\text{grad } \chi \cdot \vec{\eta}) (\text{grad } \chi \cdot \vec{\zeta}) = 0 \quad \left( \chi_x = \frac{\partial \chi}{\partial x}, \dots \right). \quad (15)$$

Thus, the system of equations determining the deformed state is kinematically determinate and belongs to the hyperbolic type. The characteristic surfaces are orthogonal to the directions of the principal strain rates.

Let us turn to the study of the equations determining the stress field. Writing condition (3) in invariants, we obtain

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0,$$

$$\varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_z \varepsilon_x - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{zx}^2 = -\kappa^2, \quad (16)$$

$$\varepsilon_x \varepsilon_y \varepsilon_z + 2\varepsilon_{xy} \varepsilon_{yz} \varepsilon_{zx} - \varepsilon_x \varepsilon_{yz}^2 - \varepsilon_y \varepsilon_{zx}^2 - \varepsilon_z \varepsilon_{xy}^2 = 0.$$

In other words, the condition for complete hardening is the vanishing of the first and third invariants of the strain-rate tensor, as well as the equality of the second invariant to a certain constant. Using condition (16) as the hardening potential, we obtain

$$ds_x = d\lambda_1 + d\lambda_2 \varepsilon_x + d\lambda_3 (\varepsilon_y \varepsilon_z - \varepsilon_{yz}^2),$$

$$d\tau_{xy} = d\lambda_2 \varepsilon_{xy} + d\lambda_3 (\varepsilon_{yz} \varepsilon_{zx} - \varepsilon_z \varepsilon_{xy}). \quad (xyz) \quad (17)$$

Next, to the 6 relations (17) one must add the condition

$$s_x + s_y + s_z = 0. \quad (18)$$

We write the equilibrium equations in the form

$$\frac{\partial \sigma}{\partial x} + \frac{\partial s_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (xyz). \quad (19)$$

The system of 10 equations (17), (18), (19) contains 10 unknowns: 7 components  $\sigma, s_x, \dots, \tau_{xy}, \dots$  plus 3 multipliers  $d\lambda_1, d\lambda_2, d\lambda_3$ . From (17) and (18) we find that  $d\lambda_1 = \frac{1}{3}\kappa^2 d\lambda_3$ ; therefore relations (17) can be rewritten in the form

$$ds_x = d\lambda_2 \varepsilon_x + d\lambda_3 \left( \varepsilon_y \varepsilon_z - \varepsilon_{yz}^2 + \frac{\kappa^2}{3} \right),$$

$$d\tau_{xy} = d\lambda_2 \varepsilon_{xy} + d\lambda_3 (\varepsilon_{yz} \varepsilon_{zx} - \varepsilon_z \varepsilon_{xy}). \quad (xyz) \quad (20)$$

Eliminating from the 6 equations (20) the 2 unknown quantities  $d\lambda_2, d\lambda_3$ , we obtain 4 equations

$$\begin{aligned} & \frac{1}{\Delta_1} \left[ ds_x (\varepsilon_{xy} \varepsilon_{xz} - \varepsilon_x \varepsilon_{yz}) - d\tau_{yz} \left( \varepsilon_y \varepsilon_z - \varepsilon_{yz}^2 + \frac{\kappa^2}{3} \right) \right] = \\ & = \frac{1}{\Delta_2} \left[ ds_y (\varepsilon_{yz} \varepsilon_{xy} - \varepsilon_y \varepsilon_{zx}) - d\tau_{zx} \left( \varepsilon_x \varepsilon_z - \varepsilon_{xz}^2 + \frac{\kappa^2}{3} \right) \right] = \\ & = \frac{1}{\Delta_3} \left[ ds_z (\varepsilon_{xz} \varepsilon_{yz} - \varepsilon_z \varepsilon_{xy}) - d\tau_{xy} \left( \varepsilon_x \varepsilon_y - \varepsilon_{xy}^2 + \frac{\kappa^2}{3} \right) \right], \quad (21) \end{aligned}$$

$$\frac{1}{\Delta_1} [ds_x \varepsilon_{yz} - d\tau_{yz} \varepsilon_x] = \frac{1}{\Delta_2} [ds_y \varepsilon_{xz} - d\tau_{xz} \varepsilon_y] = \frac{1}{\Delta_3} [ds_z \varepsilon_{xy} - d\tau_{xy} \varepsilon_z],$$

where

$$\Delta_1 = \varepsilon_x (\varepsilon_{xy} \varepsilon_{xz} - \varepsilon_x \varepsilon_{yz}) - \varepsilon_{yz} \left( \varepsilon_y \varepsilon_z - \varepsilon_{yz}^2 + \frac{\kappa^2}{3} \right),$$

$$(xyz), (1\ 2\ 3).$$

The system of 7 equations (19), (20) for the 7 unknown stress components belongs to the hyperbolic type, and its characteristic surfaces coincide with the characteristic surfaces of the equations determining the strain-rate field.

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## REFERENCES

1. W. Prager, Trans. Soc. Rheology, **1**, 169 (1957).
2. W. Prager, J. Appl. Mech., **20**, No. 3, 317 (1953).
3. W. Koiter, Quart. Appl. Math., **11**, 350 (1953).

4. W. Prager, Ph. Hodge, Theory of Perfectly Plastic Solids, N. Y.—London, 1951.
5. R. Hill, J. Mech. Phys. Solids, **4**, No. 4 (1956); **5**, No. 1, 1 (1956).
6. D. D. Ivlev, Prikl. matem. i mekh., **22**, issue 6 (1958).

*Note: Figure translations are in progress. See original paper for figures.*

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