# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON THE $\epsilon$-ENTROPY OF CERTAIN CLASSES OF FUNCTIONS

*(Presented by Academician A. N. Kolmogorov, 26 II 1960)*

Let $T$ be a compact set in a metric space $X$, and let $H_\varepsilon(T)$ be the $\varepsilon$-entropy of the set $T$, $H_\varepsilon(T) = \lg N_\varepsilon(T)$, where $N_\varepsilon(T)$ is the smallest number of elements of $X$ forming an $\varepsilon$-net for $T$, and the logarithm is taken to base 2. We shall use the signs $\asymp$ and $\sim$ for weak and strong equivalence, respectively, and also the signs $\ll$ and $\gg$, adopted in ([1]).

Let $G$ be a finite simply connected domain of the $z$-plane with rectifiable boundary $\Gamma$, whose parametric equation is $z = z(s) = x(s) + iy(s)$, $0 \leq s \leq L$, where $s$ is the arc length on $\Gamma$, measured from some point. The belonging of the curve $\Gamma$ to the class $C_\mu^m$ means that $x(s)$ and $y(s)$ have derivatives up to order $m$ inclusive for $0 \leq s \leq L$, and the derivatives of order $m$ satisfy a Lipschitz condition in $s$ with exponent $\mu$, $0 < \mu \leq 1$.

1°. Consider the question of the $\varepsilon$-entropy of a class of functions analytic in the domain $G$ and continuous in the closed domain $\overline{G}$, with additional conditions on the character of continuity in $\overline{G}$. We define the norm of a function $f(z)$ by the equality

$$\|f\| = \max_{z \in \overline{G}} |f(z)|.$$

**Lemma 1.** *If the boundary of the domain $G$ is a rectifiable Jordan curve and $T_q$ is the class of functions analytic in the domain $G$, continuous together with their derivatives up to order $p$ inclusive in $\overline{G}$, and satisfying the conditions*

$$|f^{(k)}(z)| \leq C_k \quad (k = 0, 1, 2, \dots, p), \qquad z \in \overline{G}; \tag{1}$$

$$|f^{(p)}(z_1) - f^{(p)}(z_2)| \leq C|z_1 - z_2|^\alpha, \qquad 0 < \alpha \leq 1, \tag{2}$$

*for any points $z_1$ and $z_2 \in \Gamma$, where $C_k$ and $C$ are given constants $> 0$, then*

$$H_\varepsilon(T_q) \ll \left(\frac{1}{\varepsilon}\right)^{1/q}, \qquad q = p + \alpha. \tag{3}$$

Explaining the course of the proof, define the number $\Delta = (\varepsilon/2C)^{1/q}$ and consider the values of the arc $s_0 = 0,\ s_r = s_0 + r\Delta,\ r = 0, 1, 2, \dots, m$, where $m$ is such that $s_m \leq L,\ s_{m+1} > L$; the corresponding points on $\Gamma$ will be $z_r = z(s_r)$. Put

$$\beta_r^{(k)}(f) = \left[\frac{f^{(k)}(z_r)}{\varepsilon_k}\right], \qquad k = 0, 1, 2, \dots, p; \quad r = 0, 1, 2, \dots, m,$$

where

$$\varepsilon_k = \frac{\varepsilon}{2e\Delta^k}, \qquad [a + bi] = [a] + [b]\, i,$$

and associate with the function $f(z) \in T_q$ the matrix

$$\beta = \|\beta_r^{(k)}\|.$$

Further on we use the method applied by A. N. Kolmogorov in paper $(^1)$ (§ 5), but the estimate of the remainder term of Taylor' s formula is made by representing it as an integral over the contour $\Gamma$.

**Lemma 2.** If the boundary $\Gamma \in C_\mu^1,\ 0 < \mu \leqslant 1$ (a Lyapunov curve), then the following lower estimate holds

$$H_\varepsilon(T_q) \succcurlyeq \left(\frac{1}{\varepsilon}\right)^{1/q}, \qquad 0 < \alpha < 1. \tag{4}$$

The proof can be obtained with the aid of Theorem 1 from paper $(^2)$, using the direct theorem on best approximation in a complex domain $(^3)$ and the inverse theorem of S. N. Mergelyan $(^4)$ for a domain with a boundary of the type under consideration.

**Theorem 1.** If $\Gamma \in C_\mu^1,\ 0 < \alpha < 1$, then

$$H_\varepsilon(T_q) \asymp \left(\frac{1}{\varepsilon}\right)^{1/q}. \tag{5}$$

In the case $\alpha = 1$ the relation

$$\lg H_\varepsilon(T_q) \sim \frac{1}{q} \lg \frac{1}{\varepsilon} \tag{6}$$

is valid.

Let us note that the theorem remains valid under the condition on $\Gamma$ that is somewhat less restrictive:

$$\int_0^c \frac{j(h)}{h} |\lg h|\, dh < \infty,$$

where $j(h)$ is the modulus of continuity of the function $z'(s)$.

The following two assertions are also valid.

**Theorem 2.** If the boundary of the domain $G$ is an arbitrary smooth curve with continuously rotating tangent and $0 < \alpha \leqslant 1$, then

$$\lg H_\varepsilon(T_q) \sim \frac{1}{q} \lg \frac{1}{\varepsilon}.$$

**Theorem 3.** If the boundary $\Gamma$ of the domain $G$ is an arbitrary rectifiable Jordan curve and $0 < \alpha \leqslant 1$, then*

$$\lg H_\varepsilon(T_q) \asymp \lg \frac{1}{\varepsilon}.$$

2°. Let us consider the question of the $\varepsilon$-entropy for the class of generalized analytic functions ($^5$). Suppose the equation

$$\frac{dw}{d\bar{z}} = A(z)w + B(z)\bar{w} = 0, \tag{7}$$

is given, where $w(z) = u(x, y) + iv(x, y)$, $z = x + iy$, and

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right).$$

Such an equation represents the writing, in complex form, of the elliptic system of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = au + bv, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = cu + dv, \tag{8}$$

whose coefficients depend on $x$ and $y$. We shall say that a function $\Phi(z) \in C^m_\mu(\bar{G})$ if $\Phi(z)$ and all its partial derivatives with respect to $x$ and $y$ up to order $m$ inclusive are continuous in $\bar{G}$, and all partial derivatives of order $m$ satisfy in the closed domain $\bar{G}$ a Lipschitz condition of order–

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* The upper estimate for $H_\varepsilon(T_q)$ follows from Lemma 1; the lower estimate can be obtained in the form

$$H_\varepsilon(T_q) \geq A\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2p+2}},$$

where $A$ is a constant.

with respect to $z$ with exponent $\mu,\ 0<\mu\leqslant 1$. The norm of the function $w(z)$ is defined by the equality $\|w\|=\max\limits_{z\in\overline{G}}|w(z)|$.

**Theorem 4.** If $\Gamma\in C^p_\alpha,\ 0<\alpha<1,\ A$ and $B\in C^p_\alpha(\overline{G})$, then for the class $W_q$ of solutions $w(z)$ of equation (7) satisfying the conditions

$$|w(z)|\leqslant C_0,\qquad \left|\frac{\partial^k w}{\partial x^s\partial y^{k-s}}\right|\leqslant C_k,\qquad s=0,1,\dots,k,\quad k=0,1,\dots,p,$$

$z\in\overline{G}$, and the requirement that the Lipschitz constant with exponent $\alpha$ for each partial derivative of order $p$ of $w$ not exceed $C$ ($C_k$ and $C>0$), the relation

$$H_\varepsilon(W_q)\asymp\left(\frac{1}{\varepsilon}\right)^{1/q},\qquad q=p+\alpha, \tag{9}$$

holds. For $\alpha=1$ we have

$$\lg H_\varepsilon(W_q)\sim\frac{1}{q}\lg\frac{1}{\varepsilon}. \tag{10}$$

Let us note that the functions of the class $W_q$ satisfy equation (7) in the domain $G$ in the classical sense.

The upper estimate is based on the maximum-modulus principle for generalized analytic functions: $|f(z)|\leqslant M\max\limits_{z\in\Gamma}|f(z)|,\ z\in G$ (the constant $M$ depends only on the functions $A$ and $B$)—and on the method of A. N. Kolmogorov ($^1$). The lower estimate is based on the existence of a solution of the boundary-value problem $\operatorname{Re}w(z)=\gamma(z)$ for equation (7) with an arbitrary sufficiently smooth function $\gamma(z)$ and on known theorems ($^5$) on the dependence between the smoothness character of $\gamma(z)$ and the solution $w(z)$.

3°. Let us also consider the question of the $\varepsilon$-entropy of the class $V_\mu$ of entire analytic functions $f(z)$, whose growth satisfies the condition

$$|f(z)|\leqslant e^{\mu(r)}$$

on the circle $|z|=r$ for all $r>r_0$, where $r_0$ depends only on $\mu(r)$. Suppose that $\mu(r)\uparrow\infty$ and $r\mu'(r)\uparrow\infty$ as $r\uparrow\infty$, and that the condition $r\mu''(r)[\mu'(r)]^{-1}\geqslant d>-1$ is fulfilled for $r>r_1$; let $\|f\|=\max\limits_{|z|=R}|f(z)|$. Denote by $r=h(n)$ the solution of the equation $z\mu'(r)=n$, and by $q(t)$ the solution of the equation $n\lg h(n)=t$. In both cases it suffices to find an asymptotic solution.

**Theorem 5.** The following relation holds:

$$H_\varepsilon(V_\mu) \sim q\left(\lg\frac{1}{\varepsilon}\right)\lg\frac{1}{\varepsilon}. \tag{11}$$

If, in particular, we set $\mu(r) = \sigma r^\rho$, then we obtain the result of A. G. Vitushkin (¹)

$$H_\varepsilon(V_\mu) \sim \rho\,\frac{\left(\lg\frac{1}{\varepsilon}\right)^2}{\lg\lg\frac{1}{\varepsilon}}.$$

If we set $\mu(r) = e^{\sigma r^\rho}$, then we shall have

$$H_\varepsilon(V_\mu) \sim \rho\,\frac{\left(\lg\frac{1}{\varepsilon}\right)^2}{\lg\lg\lg\frac{1}{\varepsilon}}.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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