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Abstract

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ON A NONLINEAR BOUNDARY-VALUE PROBLEM OF RIEMANN TYPE

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1. Among works concerning nonlinear boundary-value problems in the theory of analytic functions and the closely related nonlinear singular integral equations, the main place is occupied by works devoted to general questions of the existence of a solution. The methods of investigation are the Schauder principle and the principle of contraction mappings (Banach). This determines the applicability of the results obtained to equations

$$u = \lambda A u + f, \qquad K u = \lambda A u + f$$

(K a linear operator, A a nonlinear operator) only in the case of small values of the parameter λ. At the same time, the linear Riemann and Hilbert boundary-value problems are solvable in closed form. Naturally, the question arises of finding such types of nonlinear boundary-value problems to which the methods of the theory of linear problems can be applied and whose solution, consequently, can be given in closed form.

The first investigation in this direction was undertaken by P. V. Solov' ev ([5]). He indicated particular solutions of the problems

$$a(t)[\Phi^+(t)]^2 + b(t) = \Phi^-(t), \qquad a(t)[\Phi^+(t) + b(t)]^2 = \Phi^-(t)$$

for the case $\operatorname{ind} a(t) = -2\varkappa \leq 0$.* Later V. K. Natanevich ([3,4]) carried out an investigation of the nonlinear Hilbert boundary-value problem $f(u, v) = q(s)$ for the case when $f(u, v)$ is a so-called isothermal binomial of the second or third order.

In the present work the following nonlinear boundary-value problem of Riemann type is considered:

$$[\Phi^+(t)]^n = G(t)\Phi^-(t) + g(t).$$

The solutions of this problem are expressed in terms of the solutions of the corresponding linear problem by means of an n-th root. In this connection branch points may arise, and an investigation is required of the conditions under which the latter are absent. Such an investigation constitutes the main content of the present work. In what follows we use the theory of the linear Riemann boundary-value problem (see ([1]), pp. 93-114).

2. We formulate the statement of the problem. In the plane of the complex variable there is given a simple smooth closed contour L, dividing the plane into two domains: the interior D^+ and the exterior D^-. It is required to find two functions $\Phi^+(z)$ and $\Phi^-(z)$, analytic respectively in D^+ and D^-, whose limiting values satisfy on L the relation

$$[\Phi^+(t)]^n = G(t)\Phi^-(t) + g(t), \tag{1}$$

* He also considered the problem $a(t)[\Phi^+(t)]^2 + b(t)\Phi^+(t) = \Phi^-(t)$; however, its solution is erroneous (see ([2]), p. 509).

where $G(t)$ and $g(t)$ are given functions of points of the contour, satisfying the Hölder condition, and $G(t)$ does not vanish; $n \geq 2$ is an integer. If $g(t) \equiv 0$, then we have a homogeneous problem; otherwise, a nonhomogeneous one.

By means of the substitution

$$\Phi_1^+(z) = [\Phi^+(z)]^n, \qquad \Phi_1^-(z) = \Phi^-(z) \tag{2}$$

we reduce problem (1) to the linear Riemann problem

$$\Phi_1^+(t) = G(t)\Phi_1^-(t) + g(t). \tag{3}$$

If problem (1) has a solution, then problem (3) also has a solution. Consequently, all solutions of problem (1) are to be found among the functions

$$\varphi^+(z) = \sqrt[n]{\Phi_1^+(z)}, \qquad \varphi^-(z) = \Phi_1^-(z),$$

where $\Phi_1^+(z)$ and $\Phi_1^-(z)$ are solutions of problem (3). In order that $\varphi^+(z)$ and $\varphi^-(z)$ be a solution of problem (1), it is necessary and sufficient that $\sqrt[n]{\Phi_1^+(z)}$ have no branch points in D^+, i.e., that $\Phi_1^+(z)$ either have no zeros at all or have only zeros whose orders are multiples of n.

The following is proved without difficulty.

Theorem 1. *For $\chi = \operatorname{ind} G(t) \geq 0$, the homogeneous and nonhomogeneous problems (1) are solvable; the solutions depend on $\chi + 1$ complex constants and are expressed, respectively, by the formulas:*

$$\Phi^+(z) = e^{\frac{1}{n}\Gamma^+(z)} \sqrt[n]{P_\chi(z)}, \qquad \Phi^-(z) = e^{\Gamma^-(z)} z^{-\chi} P_\chi(z); \tag{4}$$

$$\Phi^+(z) = e^{\frac{1}{n}\Gamma^+(z)} \sqrt[n]{P_\chi(z) + F^+(z)}, \qquad \Phi^-(z) = e^{\Gamma^-(z)} z^{-\chi} [P_\chi(z) + F^-(z)], \tag{5}$$

where

$$\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\ln[t^{-\chi} G(t)]}{t - z}\, dt; \qquad F(z) = \frac{1}{2\pi i} \int_L \frac{g(t) e^{-\Gamma^+(t)}}{t - z}\, dt;$$

$$P_\chi(z) = C_0 z^\chi + C_1 z^{\chi - 1} + \ldots + C_\chi$$

is a polynomial of degree χ with arbitrary complex coefficients.

For $\chi \leq -1$, the homogeneous problem is unsolvable; the nonhomogeneous problem is solvable if and only if the following conditions are satisfied:

$$1) \quad \int_L g(t) e^{-\Gamma^+(t)} t^{m-1}\, dt = 0, \quad m = 1, \ldots, -\chi - 1;$$

$$2) \quad \sqrt[n]{F^+(z)} \text{ has no branch points in } D^+.$$

If these are fulfilled, the solution can be obtained from formulas (5) with $P_\chi(z) \equiv 0$.

Now the question arises: to what extent are the coefficients of the polynomial $P_\chi(z)$, entering into the solutions (4) and (5), arbitrary for $\chi \geq 0$? In the case of a linear problem the answer is simple—the coefficients are linearly independent, i.e., they can be chosen so as to satisfy $\chi + 1$ linear conditions; for example, one may require that the solution take prescribed values at $\chi + 1$ points (indifferently in D^+ or D^-). The situation is different in the case of the nonlinear problem (1). Here the number of additional conditions that can be satisfied depends essentially on the form of the conditions themselves. Thus, for example, if one requires that at some point $z_0 \in D^+$ the function $\Phi^+(z)$ have a simple zero, then, as already noted—

as was seen above, this will be equivalent to the requirement that $P_\chi(z)$ or $P_\chi(z) + F^+(z)$ have a zero of order n at z_0.

We shall impose on the desired solution the following standard conditions (a zero of order k is counted k times):

$$\Phi^+(z_i) = 0, \qquad i = 1, \ldots, k_0^+; \quad \Phi^-(w_j) = 0, \qquad j = 1, \ldots, k_0^-; \tag{6}$$

$$\Phi^+(z_{k_0^+ + i}) = a_i, \qquad i = 1, \dots, k^+; \quad \Phi^-(w_{k_0^- + j}) = b_j, \qquad j = 1, \dots, k^-,$$

and we pose the question: how many of these conditions can a solution of problem (1), for a given index, satisfy?

In conditions (6), the zero values are singled out, so that $a_i \neq 0$ and $b_i \neq 0$.

3. Theorems answering the question posed are based on a number of auxiliary propositions, of which we shall give two principal ones.

Theorem 1. *Among the polynomials $P_m(z)$ of degree $m = nk + l$, assuming at the points $z_1, \dots, z_k, w_1, \dots, w_l$ the values $a_1, \dots, a_k, b_1, \dots, b_l$, there exist polynomials of the form*

$$C \prod_1^k (z - \alpha_r)^n \prod_1^l (z - \beta_s),$$

whose multiple zeros satisfy

$$\alpha_r \in \gamma_\delta(z_r) \quad (r = 1, \dots, k),$$

and simple zeros

$$\beta_s \in \gamma_\delta(w_s) \quad (s = 1, \dots, l); \qquad 0 < \delta < \Delta/2,$$

where Δ is the minimum distance of the points from one another, and $\gamma_\delta(z)$ is the circular δ-neighborhood of the point z.

One of the ways of proving this theorem consists in constructing an iterative process followed by a proof of its convergence. Convergence is ensured by a sufficiently large value of m.

Similarly one proves:

Theorem 2. *Let the points z_i $(i = 1, \dots, k)$, w_j $(j = 1, \dots, l)$ not lie on L. Then, among the polynomials $P_m(z)$, $m = nk + l$, satisfying the conditions*

$$P_m(z_i) + \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t - z_i}\, dt = a_i, \qquad P_m(w_j) + \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t - w_j}\, dt = b_j,$$

there exist such that

$$P_m(z) + \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t - z}\, dt =$$

$$= \prod_1^k (z - \alpha_r)^n \prod_1^l (z - \beta_s) \left[C + \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{\prod_1^k (t - \alpha_r)^n \prod_1^l (t - \beta_s)(t - z)}\, dt \right],$$

$$\alpha_r \in \gamma_\delta(z_r), \quad \beta_s \in \gamma_\delta(w_s)$$

and the square bracket does not vanish in D^+; $0 < \delta < \Delta(L)/4$, where $\Delta(L)$ is the minimum distance of the points from one another and from L.

4. The following two theorems give the answer to the question posed.

In the case of the homogeneous problem

$$[\Phi^+(t)]^n = G(t) \cdot \Phi^-(t), \qquad \operatorname{ind} G(t) = \chi \geq 0 \tag{11}$$

the following is true.

Theorem II. *If one of the inequalities is fulfilled:*

$$\text{A.} \quad k^+ + k^- \leq \left[\frac{\chi - nk_0^+ - k_0^-}{n} \right] + 1; \quad nk_0^+ + k_0^- \leq \chi;$$

$$\text{B.} \quad n(k_0^+ + k^+) + k_0^- + k^- \leq \chi,$$

then problem (1) has a solution satisfying conditions (6). If both inequalities are not satisfied, then, generally speaking, problem (1^1) has no such solution.

The last assertion can be made more precise as follows: from any prescribed conditions (6), by changing in them, if necessary, one or two values, one can obtain conditions such that there exists no solution of problem (1^1) satisfying them.

The proof is based mainly on Theorem 1. In the case of the inhomogeneous problem

$$[\Phi^+(t)]^n = G(t)\Phi^-(t) + g(t), \qquad \operatorname{ind} G(t) = \varkappa \geq 0 \tag{1^2}$$

there holds

Theorem III. If

$$\text{B. } n(k_0^+ + k^+) + k_1^- + k^- \leq \varkappa,$$

then problem (1^2) has a solution satisfying conditions (6), for an arbitrary free term $g(t)$. If B is violated, then problem (1^2), generally speaking, has no such solution.

The last assertion is made more precise as follows: from any prescribed conditions (6), by changing in them, if necessary, one or two values, one can obtain conditions such that, for some $g(t)$, there exists no solution of problem (1^2) satisfying them.

Here the basis of the proof is Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.