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Abstract

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MATHEMATICS

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ON STATIONARY SEQUENCES FORMING A BASIS

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In linear prediction of a multidimensional stationary random process $x(t) = \{x_1(t), \dots, x_n(t)\}$ (the time t takes only integer values), the question of the possibility of representing the quantities of the best prediction in the form of a series

$$h = \sum_{k=1}^{n} \sum_{t \in T} c_k(t) x_k(t) \tag{1}$$

(convergent in the mean square) in terms of the values $x_k(t)$ observed at time instants $t \in T$ is very important.

Let H denote the linear closure in the mean square of the quantities $x_k(t)$, $k = 1, \dots, n$, $-\infty < t < \infty$. As usual, we identify all random variables h that differ from one another only with probability zero, and introduce in H the scalar product $(h_1, h_2) = M h_1 \overline{h_2}$.

The question of representability of random variables $h \in H$ in the form of the series (1) is the question of when the system of quantities $\{x_k(t)\}$, $k = 1, \dots, n$, $-\infty < t < \infty$, forms a basis in the Hilbert space H. In considering this question it is natural to restrict oneself to the case when the system $\{x_k(t)\}$ is minimal, i.e. no quantity $x_k(t)$ belongs to the linear closure of the remaining quantities of this system.

Let the process $x(t)$ have spectral density $f(\lambda) = \{f_{kj}(\lambda)\}_{k,j=1,\dots,n}$. From work ([2]) (cf. ([1])) follows Theorem 1.

Theorem 1. In order that the system $\{x_k(t)\}$ be minimal, it is necessary and sufficient that

$$\int_{-\pi}^{\pi} \frac{1}{\operatorname{Sp} f(\lambda)} \, d\lambda < \infty, \tag{2}$$

where

$$\operatorname{Sp} f(\lambda) = \sum_{k=1}^{n} f_{kk}(\lambda)$$

is the trace of the spectral density $f(\lambda)$.

As is known, the system $\{x_k(t)\}$ is minimal if and only if there exists in the space a conjugate system of quantities $\{y_k(t)\}$, i.e. one such that

$$(x_k(t), y_j(s)) = \begin{cases} 1 & \text{if } k = j,\ t = s; \\ 0 & \text{if } k \neq j \text{ or } t \neq s. \end{cases} \tag{3}$$

If the conjugate system $\{y_k(t)\}$ is complete in H, then each quantity $h \in H$ is uniquely determined by the series

$$h \sim \sum_k \sum_t c_k(t) x_k(t), \tag{4}$$

where $c_k(t) = (h, y_k(t))$, and if the series in (4) converges, then its sum is precisely h.

Let

$$x_k(t) = \int_{-\pi}^{\pi} e^{i\lambda t} \Phi_k(d\lambda), \qquad k = 1, \dots, n, \tag{5}$$

be the spectral representation of the process $x(t)$. Every quantity $h \in H$ can be represented in the spectral form (5)

$$h = \int_{-\pi}^{\pi} \sum_{k=1}^{n} \varphi_k(\lambda) \Phi_k(d\lambda), \tag{6}$$

where the vector function $\varphi_\lambda = \{\varphi_1(\lambda), \dots, \varphi_n(\lambda)\}$ satisfies the condition

$$\int_{-\pi}^{\pi} (\varphi_\lambda, f_\lambda \varphi_\lambda)\, d\lambda < \infty. \tag{7}$$

Relation (6) gives an isometric correspondence between the space H and the space L^2 of vector functions φ_λ with scalar product

$$(\varphi, \varphi') = \int_{-\pi}^{\pi} (\varphi_\lambda, f_\lambda \varphi'_\lambda)\, d\lambda.$$

By virtue of the minimality condition (2), the matrix function

$$f_\lambda^{-1} = \{p_{kj}(\lambda)\}_{k,j=1,\dots,n}$$

is integrable, and therefore vector functions of the form $e^{i\lambda t} f_\lambda^{-1}\delta_k$, where $\delta_k = \{0,\dots,1,0,\dots,0\}$ is the unit vector, belong to the space L^2.

Obviously, the quantities of the conjugate system $\{y_k(t)\}$ are represented in the form

$$y_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda t}\sum_{j=1}^{n} p_{kj}(\lambda)\Phi_j(d\lambda), \tag{8}$$

whence the completeness of $\{y_k(t)\}$ in the space H follows easily; the coefficients $c_k(t)$ in the expansion (4) are the Fourier coefficients of the functions $\varphi_k(\lambda)$ occurring in the representation (6):

$$c_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\lambda t}\varphi_k(\lambda)\,d\lambda. \tag{9}$$

Following [the work] ([3]), we shall call the minimal system $\{x_k(t)\}$ **Bessel** if, for every $h \in H$,

$$\sum_k\sum_t |c_k(t)|^2 < \infty,$$

and **Hilbert** if, for any numbers $c_k(t)$,

$$\sum_k\sum_t |c_k(t)|^2 < \infty,$$

there exists an $h \in H$ with such expansion coefficients that $c_k(t) = (h, y_k(t))$.

Theorem 2. *In order that the system $\{x_k(t)\}$ be Bessel, it is necessary and sufficient that*

$$f(\lambda) \geq mI \tag{10}$$

for some $m > 0$ for almost all λ.

Proof. As is known ([3]), if the system $\{x_k(t)\}$ is Bessel, then there exists a constant C such that

$$\sum_k\sum_t |c_k(t)|^2 \leq C\|h\|^2, \tag{11}$$

which leads to the relation

$$\int_{-\pi}^{\pi} \|\varphi_\lambda\|^2 \, d\lambda \leq C \int_{-\pi}^{\pi} (\varphi_\lambda, f_\lambda \varphi_\lambda) \, d\lambda \tag{12}$$

for any vector-function $\varphi_\lambda \in L^2$, whence the assertion of the theorem follows.

The following theorem is proved analogously:

Theorem 3. *In order that the system $\{x_k(t)\}$ be a Hilbert system, it is necessary and sufficient that*

$$f(\lambda) \leq MI \tag{13}$$

for some $M < \infty$ for almost all λ.

Let us note that in the case of a Hilbert system, for any numbers $c_k(t)$, $\sum_k \sum_t |c_k(t)|^2 < \infty$, the series (4) converges (4). Thus, if the functions $\varphi_k(\lambda)$ in the spectral representation (6) of the random variable h are square-integrable, then in the case where the system $\{x_k(t)\}$ is Hilbert, the random variable h is expanded in the convergent series (4).

Recall (5) that to the best quantities $\hat{x}_k(t, \tau)$ of linear extrapolation of the unknown values $x_k(t)$, $k = 1, \dots, n$, from the known past of the process—the quantities $x_k(s)$, $k = 1, \dots, n$, $s \leq \tau$—there correspond in the space L^2 vector-functions

$$\hat{\varphi}_k(t, \tau, \lambda) = \{\hat{\varphi}_{k1}(\lambda), \dots, \hat{\varphi}_{kn}(\lambda)\},$$

the components $\hat{\varphi}_{kj}(\lambda)$ of which form the matrix $\hat{\varphi}(t, \tau, \lambda)$,

$$\varphi(t, \tau, \lambda) = e^{i\lambda t} \left[a(\lambda) - \sum_{s=0}^{t-\tau-1} e^{-i\lambda s} a_s \right] a^{-1}(\lambda), \tag{14}$$

where

$$a(\lambda) = \sum_{s=0}^{\infty} e^{-i\lambda s} a_s$$

is the boundary value of a maximal analytic matrix of class H_2,

$$a(\lambda) a^*(\lambda) = 2\pi f(\lambda). \tag{15}$$

The minimality condition (2) guarantees the square-integrability of the elements of the matrix $a^{-1}(\lambda)$; from condition (13) it follows that $\|a(\lambda)\|^2 \leq M$ almost everywhere.

The considerations stated make it possible to conclude that the elements of the matrix $\varphi(t, \tau, \lambda)$ in the case of a Hilbert system $\{x_k(t)\}$ are square-integrable.

Next, to the quantities $\hat{x}_k(t, T)$ of best interpolation of the unknown values $x_k(t)$, $k = 1, \dots, n$, $t \in T$, from the values $x_k(s)$, $k = 1, \dots, n$, $s \notin T$, observed at the remaining instants of time, there correspond in the space L^2 vector-functions whose components have the form (5)

$$e^{i\lambda t}\delta_k - \sum_{j=1}^{n} p_{kj}(\lambda) \sum_{s \in T} c_s e^{i\lambda s}, \tag{16}$$

where $p_{kj}(\lambda)$ are the elements of the matrix $f^{-1}(\lambda)$.

Summarizing what has been said, we obtain Theorem 4.

Theorem 4. *If the system $\{x_k(t)\}$ is Hilbert, then the quantities $\hat{x}_k(t, \tau)$ of best extrapolation of the unknown values $x_k(t)$, $k = 1, \dots, n$, from the known values $x_k(s)$, $k = 1, \dots, n$, $s \leq \tau$, are expressible in the form of the series (4) in these values.*

If, in addition, the condition

$$\int_{-\pi}^{\pi} \frac{1}{[\mathrm{Sp}\, f(\lambda)]^2}\, d\lambda < \infty, \tag{17}$$

is satisfied,

then, in the form of the series (4), they are expressed through the known values $x_k(s)$, $k = 1, \dots, n$, $s \in T$, and the quantities $\hat{x}_k(t, T)$ of the best interpolation of the unknown values $x_k(t)$, $k = 1, \dots, n$, $t \in T$.

Following (4), we shall say that the system $\{x_k(t)\}$ forms an **unconditional basis** if, under any permutation of the quantities $x_k(t)$, it is a basis.*

As I. M. Gelfand showed (4), the system $\{x_k(t)\}$ forms an unconditional basis if and only if it is simultaneously both a Bessel and a Hilbert system.

Combining the results obtained above, we obtain Theorem 5.

Theorem 5. *In order that the system $\{x_k(t)\}$ be an unconditional basis, it is necessary and sufficient that the condition*

$$mI \leqslant f(\lambda) \leqslant MI \tag{18}$$

hold for some $m > 0$, $M < \infty$, for almost all λ.

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* That is, the sum of the series in (4) does not change under a permutation of the terms.

Note: Figure translations are in progress. See original paper for figures.