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Abstract

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ON AN INTERPOLATION THEOREM IN OPERATOR THEORY

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In the present theorem we give one general theorem of the theory of operators in Banach spaces, which contains, as special cases, a number of facts from various branches of functional analysis.

1. Let E be a Banach space, and M a certain linear set on which a family of linear operators $T(z)$ is defined, acting from M into E and depending on the complex parameter z.

Suppose the following conditions are satisfied:

A. For every $x \in M$, the function $T(z)x$ is an entire analytic function of z with values in the space E ([1]), not identically equal to zero.

B. The function $\|T(z)x\|_E$ is bounded on every straight line parallel to the imaginary axis.

For each real α introduce on the set M the norm

$$\|x\|_\alpha = \sup_{-\infty<\tau<\infty} \|T(\alpha + i\tau)x\|_E \tag{1}$$

and complete the resulting normed space to the Banach space E_α. The family of Banach spaces E_α $(-\infty < \alpha < \infty)$ will be called an **analytic scale of spaces**. Let us indicate an important property of the norm (1). It is not difficult to verify that the norm of the analytic function $T(z)x$ is a logarithmically subharmonic function in the entire complex plane. By virtue of the theorem on three lines for a logarithmically subharmonic function, $\|x\|_\alpha$ will be a logarithmically convex function of α, i.e., the following inequality will hold: for $\alpha \leqslant \beta \leqslant \gamma$,

$$\|x\|_\beta \leqslant \|x\|_\alpha^{\frac{\gamma-\beta}{\gamma-\alpha}} \|x\|_\gamma^{\frac{\beta-\alpha}{\gamma-\alpha}} \quad (x \in M). \tag{2}$$

From inequality (2) it follows that

$$\|x\|_\beta \leqslant \frac{\gamma-\beta}{\gamma-\alpha}\, \varepsilon^{-(\gamma-\beta)} \|x\|_\alpha + \frac{\beta-\alpha}{\gamma-\alpha}\, \varepsilon^{\beta-\alpha} \|x\|_\gamma \quad (\varepsilon > 0). \tag{3}$$

Conversely, if (3) is satisfied for every $\varepsilon > 0$, then (2) follows from it.

2. **Example 1.** An important example for us of an analytic scale of spaces will be the following: the space $E = H$ is Hilbert; A is a positive self-adjoint operator acting in H; M is an everywhere dense set in H, consisting of elements on which all powers of the operator A are defined; the family of operators $T(z) = A^z$. We shall denote the corresponding scale of spaces by H_A, and the spaces themselves by H_α.

It is obvious that $H_0 = H$. If A is an unbounded positive-definite operator, then for $\alpha > 0$ one may identify H_α with the domain of definition $D(A^\alpha)$ of the power A^α of the operator A. Inequality (2) in this case coincides with the known inequality for moments ([2]).

Example 2. As a concrete example of the preceding construction, consider the Hilbert space H of complex-valued functions,

defined in n-dimensional space, with summable square of the modulus. For $f(P) \in H$ denote by $\tilde{f}(Q)$ its Fourier transform. Denote by A the operator which assigns to a function $f(P)$ the function $Af(P)$, whose Fourier transform has the form

$$\widetilde{Af}(Q) = (1 + |Q|)\tilde{f}(Q).$$

With its natural domain of definition this operator will be self-adjoint. The scale of spaces obtained by the scheme described above will be denoted by $\{W_2^\alpha\}$. For Q equal to a positive integer l, the space W_2^l will coincide with the corresponding Sobolev space ([3]), and for positive α, with the space of Slobodetskii ([4]).

Example 3. Consider continuous functions in the closure $\overline{G}$ of an n-dimensional domain G. The set M consists of functions $f(P)$ that are equal to zero in a neighborhood (each in its own) of a certain manifold Ω, lying in $\overline{G}$ and of smaller dimension (for example, Ω is a point or Ω is the boundary of the domain). On M we define the operators $T(z)f(P) = f(P)/r^z$, where r is the distance from the point P to the manifold Ω, regarded as operators acting from M into the space $E = L_p(G)$. The analytic scale of spaces constructed from these operators will be denoted by $\{L_{p,\alpha}(G, \Omega)\}$ or $\{L_{p,\alpha}\}$.

3. Generalizations. Such classical scales of spaces as spaces of functions satisfying Hölder conditions, or the spaces L_p, do not fall under the concept of an analytic scale. The concept of an analytic scale can be extended. First of all, one may abandon the requirement that the function $T(z)x$ be entire, and assume that it is analytic inside the strip $\alpha_0 < \operatorname{Re} z < \beta_0$. Under such a definition the Hölder classes C_α, with the natural norm, will form an analytic scale for $0 < \alpha < 1$. To construct the scale of spaces L_p, one should abandon the linearity of the operators $T(z)$.

Example 4. Consider measurable functions in an n-dimensional domain G. Let M denote the set of functions taking only a finite number of different values. Let E be the space $L_1(G)$. Let $T(z)f = |f|^z$. It is obvious that $T(z)f$, for

$f \in M$, will be an entire analytic function with values in $L_1(G)$. The function

$$\varphi(t) = \sup_{-\infty<\tau<\infty} \left\| |f|^{t+i\tau} \right\|_{L_1} = \left\| |f|^t \right\|_{L_1}.$$

is logarithmically convex, but does not have the properties of a norm. The function

$$[\varphi(1/\alpha)]^\alpha = \left\| |f|^{1/\alpha} \right\|_{L_1}^\alpha = \|f\|_{L_{1/\alpha}}$$

is logarithmically convex as a function of α and has the properties of a norm for $0 \leqslant \alpha \leqslant 1$.

Thus, the scale L_p is constructed from the nonlinear operator $T(z)$, analytic in z, with a subsequent change of parameter.

4. Group property. In Examples 1-3 the operators $T(z)$ form a commutative group, and therefore

$$\|T(\beta+i\sigma)x\|_\alpha = \sup_{-\infty<\tau<\infty} \|T(\alpha+i\tau)T(\beta+i\sigma)x\|_E$$

$$= \sup_{-\infty<\tau<\infty} \|T(\alpha+\beta+i(\tau+\sigma))x\|_E = \|x\|_{\alpha+\beta}.$$

In connection with this we introduce the following condition:

C. The set $M \subset E$ is invariant with respect to the operators $T(z)$. The operator $T(0)$ is the identity. For every $x \in M$ the function $T(z)x$ is analytic in every space E_α, and

$$\|T(\beta+i\sigma)x\|_\alpha \leqslant \|x\|_{\alpha+\beta}. \quad (4)$$

In Examples 1-3, condition C is satisfied; for Example 4 one can write an analogous condition, in which $\alpha+\beta$ will be replaced by $\alpha\beta$.

5. **The conjugate scale.** Let two analytic scales $\{E_\alpha\}$ and $\{E'_\alpha\}$ $(-\infty < \alpha < \infty)$ be given, constructed respectively on the sets M and M'. We shall say that the scale $\{E'_\alpha\}$ is **conjugate** to the scale $\{E_\alpha\}$ if there exist a bilinear functional (x,u), defined for $x \in M$ and $u \in M'$, and a linear correspondence $\alpha \longleftrightarrow \alpha^*$ such that

$$\|x\|_{E_\alpha} = \sup_{u\in M'} \frac{|(x,u)|}{\|u\|_{E'_{\alpha^*}}}. \quad (5)$$

It follows from (5) that each space E'_{α^*} may be regarded as embedded in the space E^*_α, conjugate to E_α, and moreover $\|u\|_{E^*_\alpha} \leq \|u\|_{E'_{\alpha^*}}$.

In Examples 1, 2, and 4 the scales are conjugate to themselves, with $\alpha^* = -\alpha$ in Examples 1 and 2, and $\alpha^* = 1-\alpha$ in Example 4. In Example 3, the scale conjugate to $\{L_{p,\alpha}\}$ is the scale $\{L_{-p',-\alpha}\}$ $\left(\frac{1}{p'}+\frac{1}{p}=1\right)$.

6. **Interpolation theorem 1.** Let $\{E_\alpha\}$ and $\{E_{\bar\alpha}\}$ be two analytic scales, and suppose there exists a scale $\{E'_\alpha\}$ conjugate to $\{E_{\bar\alpha}\}$. Assume that the scales $\{E_\alpha\}$ and $\{E'_\alpha\}$ satisfy condition C. Let an operator Q be defined on the set M corresponding to the scale $\{E_\alpha\}$, such that for some α, β and $\bar\alpha, \bar\beta$

$$\|Qx\|_{\bar\alpha} \le K_1 \|x\|_\alpha, \qquad \|Qx\|_{\bar\beta} \le K_2 \|x\|_\beta \quad (x \in M). \tag{6}$$

Denote $\alpha(\mu) = \mu\beta + (1-\mu)\alpha, \quad \bar\alpha(\mu) = \mu\bar\beta + (1-\mu)\bar\alpha$. Then

$$\|Qx\|_{\bar\alpha(\mu)} \le K_1^{1-\mu} K_2^{\mu} \|x\|_{\alpha(\mu)}. \tag{7}$$

The method of proof of the theorem is close to the method of proof of M. Riesz' s theorem proposed by Calderón and Zygmund ([5]), and consists in applying the three-lines theorem to the analytic function

$$\Phi(z) = (QT(z(\beta-\alpha))x, S^*(z(\bar\beta^* - \bar\alpha^*))y),$$

where $T(z)$ and $S^*(z)$ are the operators corresponding to the scales $\{E_\alpha\}$ and $\{E'_{\bar\alpha}\}$.

Remark. Analysis of the proof of the theorem shows that it admits a more general formulation. Without giving it, we note that this formulation is such that it includes the case when one of the scales, or both, are scales $\{L_p\}$.

We give some consequences of Theorem 1.

Concrete interpolation theorems. In the case when both scales are scales $\{L_p\}$, Theorem 1 coincides with the well-known theorem of M. Riesz.

Let now the scale $\{E_\alpha\} = \{L_{p,\alpha}\}$, and $\{E_{\bar\alpha}\} = \{L_{1/\bar\alpha}\}$. Then, for example, we arrive at the theorem:

Let the operator Q be a bounded operator acting in the space L_p with norm $\|Q\|$. If the operator Q maps some space $L_{p,\beta}$ $(\beta > 0)$ into the space L_q $(q > p)$ and $\|Qf\|_{L_q} \le C\|f\|_{L_{p,\beta}}$, then it maps each space L_r $(p < r < q)$ into the space $L_{p,\gamma}$, where $\gamma = \dfrac{r-p}{q-p}\beta$ and $\|Af\|_{L_r} \le C^{\gamma/\beta}\|A\|^{1-\gamma/\beta}\|f\|_{L_{p,\gamma}}$.

Embedding theorems. We shall say that a Banach space E_2 is **embedded** in the space E_1 if $E_2 \subset E_1$ and $\|x\|_{E_1} \le C\|x\|_{E_2}$ $(x \in E_2)$. Applying Theorem 1 to the case when the operator Q is the identity operator, one can obtain a number of embedding theorems. From the embedding theorem of S. L. Sobolev ([3]) it follows that the space W_2^l for an integer $l < n/2$ is embedded in the space $L_{1/\bar\alpha}$, where $\bar\alpha = \frac{1}{2} - l/n$. On the other hand,

sides, the spaces W_2^0 and L_2 coincide. Consequently: for all positive $\alpha < n/2$, the space W_2^α is embedded in the space $L_{\frac{2n}{n-\alpha}}$.

Similarly, using the results of ([6]), one can obtain, for example, the theorem:

For all positive $\alpha < n/2$, the space W_2^α is embedded in the space $L_{2,\alpha}(G, P_0)$, where P_0 is any point of the domain G. The norm of a function in $L_{2,\alpha}(G, P_0)$ is estimated in terms of the norm of the function in W_2^α, uniformly with respect to P_0.

One can prove a number of similar assertions concerning embeddings of W_2^α into spaces $L_{p,\beta}(G, \Omega)$ with various p, β and manifolds Ω.

Theorems on fractional powers of operators. If in the general theorem one takes for the scale $\{E_\alpha\}$ the scale H_A of Example 1, then we obtain:

Theorem 2. Let $\{E_\alpha\}$ be an analytic scale whose conjugate satisfies condition C. Let $E_0 = H$, and let A be an unbounded positive self-adjoint operator acting in H. If the domain of definition $D(A)$ of the operator A is contained in the space $E_{\bar\beta}$ and $\|x\|_{E_{\bar\beta}} \le C\|Ax\|_H$, then the domain of definition $D(A^\mu)$ of the operator A^μ $(0 \le \mu \le 1)$ is contained in the space $E_{\mu\bar\beta}$, and

$$\|x\|_{E_{\mu\bar\beta}} \le C^\mu \|A^\mu x\|_H.$$

If for the scale $\{E_\alpha\}$ one takes the scale L_p, then we obtain the theorem of M. A. Krasnosel' skii with the refinement given in ([6]). If one sets $E_\alpha = W_2^\alpha$, then we obtain a new theorem which, together with one of the embedding theorems formulated above, gives a refinement of results obtained by us with V. P. Glushko in ([7]). A new theorem is obtained likewise if one takes $E_\alpha = L_{p,\alpha}$. Finally, if for the scale $\{E_\alpha\}$ one takes the scale H_B, constructed from some positive operator B, then we arrive at the well-known inequality of E. Heinz ([8]): if $\|Bx\| \le \|Ax\|$, then $\|B^\mu x\| \le \|A^\mu x\|$. We note that from the interpolation theorem there also follows the following assertion of E. Heinz, with T. Kato' s refinement ([9]): if the operator Q is such that $\|Qx\| \le \|Ax\|$ and $\|Q^* x\| \le \|Bx\|$, then $|(Qx, y)| \le \|A^\mu x\|\,\|B^{1-\mu} y\|$ $(0 \le \mu \le 1)$. Here the operator Q is regarded as an operator acting from the spaces of the scale H_A into the spaces of the scale H_B. Inequalities (6) are then satisfied for the indices $\bar\alpha = -1$, $\bar\beta = 0$, $\alpha = 0$, $\beta = 1$. The assertion of Heinz then follows from (7).

7. Complete continuity.

Theorem 3. If, under the conditions of Theorem 1, the operator Q is completely continuous as an operator from E_α to $E_{\bar\alpha}$, then it is completely continuous as an operator from $E_{\alpha(\mu)}$ to E_γ, where γ is any number between $\bar\alpha$ and $\alpha(\mu)$.

Theorem 3 follows directly from Theorem 1 and inequality (2).

8. Uniqueness.

Theorem 4. Suppose two Banach spaces F and G are given. If there exists an analytic scale $\{E_\alpha\}$, whose conjugate has property C, such that $E_0 = F$ and $E_1 = G$, then such a scale on the interval $[0, 1]$ is uniquely determined.

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Note: Figure translations are in progress. See original paper for figures.