MATHEMATICS

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Abstract

Full Text

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APPROXIMATION OF FUNCTIONS DIFFERENTIABLE IN THE WEYL SENSE BY VALLEE-POUSSIN SUMS

(Presented by Academician A. N. Kolmogorov on 27 XI 1959)

1. Let the function $f(x)$ be summable and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

Let $r > 0$ and let α be a real number. If the series

$$\sum_{k=1}^{\infty} k^r \left[a_k \cos\left(kx + \frac{\alpha\pi}{2}\right) + b_k \sin\left(kx + \frac{\alpha\pi}{2}\right)\right]$$

is the Fourier series of some summable function, which we shall denote by $f_\alpha^r(x)$, and if $\|f_\alpha^r(x)\| \leq 1$ almost everywhere, then we shall say that $f(x) \in W_\alpha^r$ (see ([1])). For $\alpha = r$, $f_\alpha^r(x)$ is the derivative (in the Weyl sense) of order r of the function $f(x)$, and we obtain the class W^r; for $\alpha = r - 1$ we obtain the class $\overline{W}^r$ of functions conjugate to functions of the class W^r.

With the aid of the matrix $\{\lambda_{n,k}\}$, $n, k = 1, 2, ...,$ $\lambda_{n,k} = 0$ for $k \geq n$, we associate with each function $f(x) \in W_\alpha^r$ the polynomial

$$u_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \lambda_{n,k}(a_k \cos kx + b_k \sin kx)$$

and, as $n \to \infty$, find asymptotic formulas for the upper bounds

$$U_n(W_\alpha^r) = \sup_{f \in W_\alpha^r} \|f(x) - u_n(f, x)\|_C. \quad (1)$$

If the function $\tau(u) = \tau(u, n)$ is defined for $u = k/n$ by the equalities

$$\tau\left(\frac{k}{n}\right) = (1-\lambda_{n,k})\left(\frac{k}{n}\right)^{-r} \qquad (k=1,2,...),$$

is continuous, and

$$A(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\alpha\pi}{2}\right)du\right|dt < \infty, \tag{2}$$

then, for $f(x) \in W_\alpha^r$,

$$f(x) - u_n(f,x) = \frac{1}{\pi n^r}\int_{-\infty}^{\infty} f_\alpha^r\left(x+\frac{t}{n}\right)\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\alpha\pi}{2}\right)du\,dt. \tag{3}$$

Representation (3) for integral α was obtained by B. Nadem (2).
From (3) we find that, as $n \to \infty$,

$$U_n(W_\alpha^r) = A(\tau)\frac{1}{n^r} + O\left(\frac{1}{n^r}a_n(\tau)\right), \tag{4}$$

where

$$a_n(\tau) = \int_{I_n}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\alpha\pi}{2}\right)du\right|dt, \tag{5}$$

$$I_n = \left(-\infty, -\frac{n\pi}{2}\right)\cup\left(\frac{n\pi}{2},\infty\right). \tag{6}$$

2. Let $s_n(f,x)$, $n=0,1,2,...$, be the partial sums of order n of the Fourier series of the function $f(x) \in W_\alpha^r$. The polynomials

$$v_{n,m}(f,x) = \frac{1}{m}\sum_{k=n-m}^{n-1} s_k(f,x) \qquad (m=1,2,\dots,n;\ n=1,2,...) \tag{7}$$

are called the de la Vallée-Poussin sums of the function $f(x)$.
The upper bounds (1) in approximation by de la Vallée-Poussin sums will be denoted by $V_{n,m}(W_\alpha^r)$. The asymptotic behavior of $V_{n,m}(W_\alpha^r)$ as $n \to \infty$ is determined by us under the assumption that $\lim\frac{m}{n}$ exists and is equal to θ, $0 \leqslant \theta \leqslant 1$.

Theorem 1. For $V_{n,m}(W_\alpha^r)$, as $n \to \infty$, the following asymptotic formulas hold:

1) If $\theta = 0$, then

$$V_{n,m}(W_\alpha^r) = \frac{4}{\pi^2}\frac{1}{n^r}\log\frac{n}{m} + O\left(\frac{1}{n^r}\right). \tag{8}$$

2) If $0 < \theta < 1$, then

$$V_{n,m}(W_\alpha^r) = A(\tau_{1-\theta})\frac{1}{n^r} + O\left(\frac{1}{n^{r+1}}\right) + O\left(\frac{\varepsilon_n}{n^r}\right), \tag{9}$$

where

$$\tau_{1-\theta}(u) = \begin{cases} 0, & \text{for } 0 \leqslant u \leqslant 1-\theta, \\ \dfrac{u-(1-\theta)}{\theta}\, u^{-r}, & \text{for } 1-\theta \leqslant u \leqslant 1, \\ u^{-r}, & \text{for } 1 \leqslant u < \infty, \end{cases}$$

and

$$\varepsilon_n = \left|\frac{m}{n} - \theta\right| \log\frac{1}{|m/n-\theta|} \quad \text{for } \frac{m}{n} \neq \theta; \qquad \varepsilon_n = 0 \quad \text{for } \frac{m}{n} = \theta.$$

3) If $\theta = 1$ and $0 < r < 1$, then

$$V_{n,m}(W_\alpha^r) = A(\tau_{1,r})\frac{1}{n^r} + O\left(\frac{(n-m+1)^{1-r}}{n}\right), \tag{10}$$

where

$$\tau_{1,r}(u) = \begin{cases} u^{1-r}, & \text{for } 0 \leqslant u \leqslant 1, \\ u^{-r}, & \text{for } 1 \leqslant u < \infty. \end{cases}$$

4) If $\theta = 1$ and $r = 1$, then

$$V_{n,m}(\overline{W}_\alpha^1) = \frac{2}{\pi}\left|\sin\frac{\alpha\pi}{2}\right|\frac{1}{n}\log\frac{n}{n-m+1} + O\left(\frac{1}{n}\right). \tag{11}$$

If, however, $\left|\sin\frac{\alpha\pi}{2}\right| = 0$, then for $m = n$

$$V_{n,n}(\overline{W}^1) = A(\tau_{1,1})\frac{1}{n} + O\left(\frac{1}{n^2}\right), \tag{12}$$

and for $n - m \to \infty$

$$V_{n,m}(\overline{W}^1) = 2A(\tau_{1,1})\frac{1}{n} + O\left(\frac{1}{n}\sqrt{\frac{n}{n-m}}\log\frac{n}{n-m}\right) + O\left(\frac{1}{n(n-m)}\right), \tag{13}$$

where

$$\tau_{1,1}(u) = \begin{cases} 1, & \text{for } 0 \le u \le 1, \\ u^{-1}, & \text{for } 1 \le u < \infty. \end{cases}$$

5) If $\theta = 1$ and $r > 1$, then:

in the case $n - m = p \to \infty$,

$$V_{n,m}(W_\alpha^r) = A(\tau_{1,r}) \left[\frac{1}{n} + \frac{p}{n^2} + \cdots + \frac{p^{r-2}}{n^{r-1}} \right] + O\left(\frac{1}{n^r} \right) + O\left(\frac{1}{np^r} \right), \tag{14}$$

where

$$\tau_{1,r}(u) = \begin{cases} 0, & \text{for } 0 \le u \le 1, \\ (u-1)u^{-r}, & \text{for } 1 \le u < \infty; \end{cases}$$

in the case $n - m = p$ fixed, $p \ge 1$,

$$V_{n,m}(W_\alpha^r) = \frac{1}{\pi} \sup_{f \in W_\alpha^r} \left| \int_{-\infty}^{\infty} f_\alpha^r(t) \int_p^\infty \frac{u-p}{u^r} \cos\left(ut + \frac{\alpha\pi}{2} \right) du \, dt \right| \times$$
$$\times \left[\frac{1}{n} + \frac{p}{n^2} + \cdots + \frac{p^{r-2}}{n^{r-1}} \right] + O\left(\frac{1}{n^r} \right); \tag{15}$$

in the case $m = n$

$$V_{n,n}(W_\alpha^r) = \sup_{f \in W_\alpha^r} |\tilde{f}'(x)| \frac{1}{n} + O\left(\frac{1}{n^r} \right). \tag{16}$$

For approximation by Fourier sums ($m = 1$), formula (8) for the classes W^r was obtained by A. N. Kolmogorov ([3]) (for integer r) and by V. T. Pinkevich ([4]) (for all $r > 0$); for the classes $\overline{W}^r$, by S. M. Nikol' skii ([5,6]); for the classes W_α^r, by A. V. Efimov ([7]). For $m = o(n)$, formula (8) for the classes W^r and $\overline{W}^r$ was obtained by A. F. Timan ([8]).

For approximation by Fejér sums ($m = n$), formula (11) for the class W^1 belongs to S. M. Nikol' skii ([9]), and formula (12) to S. B. Stechkin (see ([10])). Formula (16) for the classes W^r, $\overline{W}^r$, and W_α^r for integer α was obtained by S. M. Nikol' skii ([11,6]) and B. Nagy ([12,2]). For the value of the upper bound appearing on the right-hand side of formula (16), see ([1,13]).

The order of decrease of the quantities $V_{n,m}(W_\alpha^r)$ for $0 < \theta < 1$ and $V_{n,n}(W_\alpha^r)$ for $0 < r \le 1$ was also known.

For the classes W^r and $\overline{W}^r$ with integer r, Theorem 1 (without formula (16)) was published by the author ([10]); here the integrals entering the constants $A(\tau)$ are given in ([10]) in transformed form. There an asymptotic formula for $V_{n,m}(\overline{W}^1)$ in the case $n-m$ fixed, not considered in Theorem 1, is also indicated.

3. If in the definition of W^r_α given above we put $r=0$, then we obtain the classes of functions W^0_α.

If, for the continuous function $\varphi(u)=\varphi(u,n)$,

$$\varphi\left(\frac{k}{n}\right)=\lambda_{n,k} \qquad (k=1,2,...) \tag{17}$$

and $A(\varphi)<\infty$, then for $f(x)\in W^0_\alpha$

$$u_n(f,x)=\frac{1}{\pi}\int_{-\infty}^{\infty} f^0_\alpha\left(x+\frac{t}{n}\right)\int_0^{\infty}\varphi(u)\cos\left(ut+\frac{\alpha\pi}{2}\right)du\,dt, \tag{18}$$

$$U_n(W^0_\alpha)=\sup_{f\in W^0_\alpha}\|u_n(f,x)\|_C=A(\varphi)+O(a_n(\varphi)). \tag{19}$$

Representation (18) for integral α was obtained by B. Nagy [(2)].
For the de la Vallée Poussin sums, the upper bounds (19) will be denoted by $V_{n,m}(W^0_\alpha)$.

Theorem 2. If $|\alpha|\leqslant 1$ and $n\to\infty$,

$$V_{n,m}(W^0_\alpha)=\left(\frac{4}{\pi^2}\cos\frac{\alpha\pi}{2}+\frac{2}{\pi}\alpha\sin\frac{\alpha\pi}{2}\right)\log\frac{n}{m}+\frac{2}{\pi}\left|\sin\frac{\alpha\pi}{2}\right|\log m+O(1). \tag{20}$$

If $\alpha=0$ and $\dfrac{m}{n}\to\theta,\ 0<\theta\leqslant 1$,

$$V_{n,m}(W^0_0)=A(\varphi_{1-\theta})+O(\varepsilon_n), \tag{21}$$

where

$$\varphi_{1-\theta}(u)=\begin{cases}1, & \text{for } 0\leqslant u\leqslant 1-\theta,\\ \dfrac{u-(1-\theta)}{\theta}, & \text{for } 1-\theta\leqslant u\leqslant 1,\\ 0, & \text{for } 1\leqslant u<\infty.\end{cases}$$

Formula (20) for $\alpha=1$ was communicated to the author by S. B. Stechkin. Formula (21) belongs to S. M. Nikol' skii [(14)] (see also [(10)]).

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Note: Figure translations are in progress. See original paper for figures.