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Abstract

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ON FORCED CONVECTION IN A PLANE LAYER IN THE PRESENCE OF AXIAL SYMMETRY

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In connection with certain technical problems, it is of interest to study solutions of the equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1-2\nu}{r}\frac{\partial U}{\partial r} - \frac{\partial U}{\partial t} + F(r,t) = 0, \tag{1}$$

which differs from the heat-conduction equation with continuously distributed sources by the presence of the convective term $-\frac{2\nu}{r}\frac{\partial U}{\partial r}$. If $\nu > 0$, then on the axis $r = 0$ there is a source of liquid with constant flow rate. If $\nu < 0$, then the axis $r = 0$ is a sink. The cases $\nu > 0$ and $\nu \leqslant 0$ are fundamentally different. If the axis $r = 0$ is a source of liquid ($\nu > 0$), the heat sources are distributed continuously and their density on the axis $r = 0$ is equal to zero, then it is physically evident that on the axis $r = 0$ the temperature U may be prescribed in advance. If, however, the axis $r = 0$ is a sink or convective heat transfer is absent, then the quantity $U(0,t)$ cannot be prescribed in advance. Thus, for $\nu > 0$, for equation (1) one may pose the boundary-value problem

$$U(r,0) = \varphi(r), \qquad U(0,r) = f(t). \tag{2}$$

The purpose of the present note is to study this boundary-value problem.

§ 1. Consider the Cauchy problem

$$\frac{\partial^2 V}{\partial r^2} + \frac{1-2\nu}{r}\frac{\partial V}{\partial r} = \frac{\partial V}{\partial t}, \qquad 0 < r < \infty; \qquad V(r,0) = \varphi(r), \quad 0 < r < \infty, \tag{1,1}$$

under the additional boundedness condition for $r^{-\nu}V(r,t)$ when $0 \leqslant r < \infty$.

We seek V in the form

$$V(r,t) = \int_0^\infty a(\lambda, r) e^{-\lambda^2 t} \lambda \, d\lambda. \tag{1,2}$$

Then (1,1) will be formally satisfied if $a = a(\lambda) r^\nu J_\nu(\lambda r)$, where $J_\nu(z)$ is the Bessel function of the first kind of order ν. Substituting this expression into (1,2), requiring the initial condition to hold, and using formally the Hankel transform for $J_\nu(z)$, we find that

$$a(\lambda) = \int_0^\infty \rho^{1-\nu} \varphi(\rho) J_\nu(\lambda\rho) \, d\rho. \tag{1,3}$$

Substituting (1,3) into (1,2), changing the order of integration, and using Weber' s second exponential integral ([1]), we find

$$V(r,t) = \int_0^\infty \varphi(\rho) E(r,\rho,t) \rho \, d\rho, \tag{1,4}$$

where it is put

$$E(r,\rho,t) = \left(\frac{r}{\rho}\right)^\nu \frac{e^{-(r^2+\rho^2)/4t}}{2t} I_\nu\left(\frac{r\rho}{2t}\right); \tag{1,5}$$

here $I_\nu(z)$ is the Bessel function of imaginary argument. Put

$$L_{r,t} = \frac{\partial^2}{\partial r^2} + \frac{1-2\nu}{r}\frac{\partial}{\partial r} - \frac{\partial}{\partial t}; \qquad L^*_{r,t} = \frac{\partial^2}{\partial r^2} + \frac{1+2\nu}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial t}. \tag{1,6}$$

A calculation shows that

$$L_{r,t}[E(r,\rho,t-\tau)] = L^*_{\rho,\tau}[E(r,\rho,t-\tau)] \equiv 0$$

for $r \neq \rho$, $t \neq \tau$, and that the operators $L_{r,t}$ and $L^*_{r,t}$ are adjoint. From $L_{r,t}E = 0$ it follows, obviously, that V, defined according to (1,4), is indeed, for $r > 0$ and $t > 0$, a solution of equation (1,1), since differentiation under the integral sign is legitimate.

The fulfillment of the initial condition follows from the following filtering property of E: for every piecewise-continuous function $\varphi(r)$ growing as $r \to \infty$ no faster than Ke^{cr^2}, where $K > 0$, $c > 0$ are arbitrary constants, and for every $\nu > -1/2$,

$$\lim_{t\to+0} \int_0^\infty \varphi(\rho) E(r,\rho,t) \rho \, d\rho = \frac{\varphi(r+0) + \varphi(r-0)}{2}, \qquad r > 0. \tag{1,7}$$

The validity of (1,7) is an obvious consequence of the asymptotic expression

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 + O\left(\frac{1}{z}\right)\right] \tag{1,8*}$$

and the rough estimate *

$$0 < E(r,\rho,t) < a_\nu \left(\frac{r^2}{4t}\right)^\nu \frac{e^{-(r-\rho)^2/4t}}{2t}, \qquad \nu > -1/2, \quad 0 \le r < \infty, \quad t > 0; \tag{1,8}$$

here $a_\nu = \max\{\Gamma^{-1}(\nu+1); \Gamma^{-1}(\nu+2)\}$.

Thus V, defined according to (1,4), is indeed a solution of the Cauchy problem (1,1). But this means that E is the temperature-influence function of an instantaneous cylindrical heat source, and therefore E may be taken as a fundamental solution of equation (1,1).

§ 2. Consider the boundary-value problem (1), (2). From Green's formula for the adjoint operators $L_{\rho,\tau}$ and $L^*_{\rho,\tau}$, the equalities $L_{\rho,\tau}[U(\rho,\tau)] \equiv 0$, $L^*_{\rho,\tau}[E(r,\rho,t-\tau)] = 0$, and the filtering property of E, it follows that, if u and $\partial u/\partial r$ satisfy the usual conditions at infinity and U is a solution of problem (1), (2), then

$$U(r,t) = 2\nu \int_0^t f(\tau)E(r,0,t-\tau)\,d\tau + \int_0^\infty \varphi(\rho)E(r,\rho,t)\rho\,d\rho +$$
$$+ \int_0^t d\tau \int_0^\infty F(\rho,\tau)E(r,\rho,t-\tau)\rho\,d\rho \equiv U_1 + U_2 + U_3. \tag{2,1}$$

In deriving (2,1) the limit

$$\lim_{\rho\to 0} \rho\,\frac{\partial U}{\partial r}\,E(r,\rho,t-\tau) = 0 \qquad \text{for } r > 0,\ t-\tau \ge 0 \tag{2,2}$$

was used.

* For $\mu > -1/2$, $\Gamma^{-1}(1+\mu)$ decreases monotonically. Therefore, for $z > 0$

$$0 < \left(\frac{z}{2}\right)^{-\nu} I_\nu(z) < a_\nu \sum_{m=0}^\infty \left(\frac{z}{2}\right)^{2m} / m!^2 < a_\nu \left[\sum_{m=0}^\infty \left(\frac{z}{2}\right)^m / m!\right]^2 = a_\nu e^{2z}.$$

We note that

$$E(r,0,t-\tau) = z^{\nu} e^{-z}/2\Gamma(\nu+1)(t-\tau), \qquad z = r^2/4(t-\tau). \tag{2.3}$$

We shall show that if $f(t)$ and $F(r,t)$ are differentiable for $0 \le r < \infty$, $0 < t \le T$, and $F(0,t) = 0$, then $U(r,t)$, defined according to (2.1), is a solution of problem (1), (2), satisfying conditions (2.2)*.

Let $r \ge r_0 > 0$, $t > 0$. Then it is obvious that $L_{r,t}[U_i] \equiv 0$ $(i = 1,2)$. The equality $L_{r,t}(U_3) + F(r,t) \equiv 0$ is proved in the same way as for the ordinary heat equation without convection (2). Thus the question reduces to verifying the fulfillment of conditions (2) and (2.2).

First of all, note that from the equality

$$2\nu \int_0^t E(r,0,t-\tau)\, d\tau = \frac{1}{\Gamma(\nu)} \int_{r^2/4t}^{\infty} e^{-z} z^{\nu-1}\, dz, \qquad t > 0, \tag{2.4}$$

it follows in the usual way that

$$\lim_{r\to 0} U_1 = \lim_{r\to 0} 2\nu \int_0^t f(\tau) E(r,0,t-\tau)\, d\tau = f(t), \qquad t > 0. \tag{2.5}$$

It is further obvious that for $\nu > 0$, $\lim_{r\to 0} U_2 = 0$. Finally, the equality

$$\lim_{r\to 0} U_3 = \lim_{r\to 0} \int_0^t d\tau \int_0^{\infty} F(\rho,\tau) E(r,\rho,t-\tau) \rho\, d\rho \equiv 0 \tag{2.6}$$

follows from the following considerations. From estimate (1.8) it follows that for any $\delta > 0$ and $|F(r,t)| < Ke^{cr^2}$, $K > 0$, $c > 0$,

$$\lim_{r\to 0} \int_0^t d\tau \int_{\delta}^{\infty} F(\rho,\tau) E(r,\rho,t-\tau)\rho\, d\rho = 0, \qquad 0 < t < \frac{1}{4c}. \tag{2.7}$$

At the same time it is easy to verify the boundedness of the integral

$$\int_0^t d\tau \int_0^{\infty} E(r,\rho,t-\tau)\rho\, d\rho \quad \text{for } 0 \le r < \infty,\ 0 < t < \infty,\ \nu > 0. \tag{2.8}$$

But this, in view of $E > 0$ and the mean value theorem, means that

$$j \equiv \left| \int_0^t d\tau \int_0^{\delta} F(\rho,\tau) E(r,\rho,t-\tau)\rho\, d\rho \right| < |F(\bar\rho,\bar t)|\, M, \tag{2.9}$$

where $M > 0$ is some constant and $0 < \bar{t} < t;\ 0 < \bar{\rho} < \delta$. Hence, from the continuity of $F(r,t)$ for $0 \le r < \infty$ and $F(0,t) = 0$, the validity of (2.6), and together with it also $\lim_{r\to 0} U_3 = 0$, obviously follows.

The fulfillment of the initial condition follows from the obvious equalities $U_i(r,0) = 0$ $(i = 1,3)$ and the filtering property of E for continuous φ.

We shall prove that, under the assumptions made, conditions (2.2) are fulfilled. Indeed, it is easy to see that near $r = 0$

$$\frac{\partial U_1}{\partial r}\begin{cases} O(r^{2\nu-1}) + O(r), & \text{for } \nu > 0,\ \nu \ne \frac{1}{2},\ t \ge t_0 > 0; \\ O(r\ln r), & \text{for } \nu = \frac{1}{2},\ t \ge t_0 > 0; \end{cases} \tag{2.10}$$

$$\frac{\partial U_2}{\partial r} = O(r^{2\nu-1}) \quad \text{for } \nu > 0,\ t \ge t_0 > 0. \tag{2.11}$$

* The differentiability requirements on f and F can obviously be weakened, for example, to Hölder conditions.

(2.11) is an obvious consequence of the equality $(z^\nu I_\nu(z))' = I_{\nu-1}(z)z^\nu$ and of the definition of E, while (2.10) follows from the definition of U_1 by means of the substitution $r^2/4(t-\tau) = z$ and differentiation with respect to r, followed by application of the mean-value theorem.

Consider $\partial U_3/\partial r$. Let

$$\psi(r,t) = \int_0^t d\tau \int_0^\infty \frac{\partial}{\partial r}E(r,\rho,t-\tau)F(\rho,\tau)\rho\,d\rho,$$

$$\psi_0(r,t) = \int_0^t d\tau \int_0^\infty \frac{\partial}{\partial r}E(r,\rho,t-\tau)\rho\,d\rho. \tag{2.12}$$

From the definition (1.5) it follows that

$$\frac{\partial E}{\partial r} = \left(\frac{r}{\rho}\right)^\nu \frac{e^{-(r^2+\rho^2)/4(t-\tau)}}{4(t-\tau)^2}\left(\rho I_{\nu-1}\left[\frac{r\rho}{2(t-\tau)}\right] - rI_\nu\left[\frac{r\rho}{2(t-\tau)}\right]\right). \tag{2.13}$$

Hence, and from (1.8*), (1.8), it follows that for $0 < \delta \le r < \infty,\ 0 < t \le T$ the integral $\psi(r,t)$ converges uniformly and, consequently, $\psi = \partial U_3/\partial r$. Consider ψ_0. Using Weber' s first exponential integral ([1]), we find

$$\psi_0 = \frac{r}{2\Gamma(\nu)}\int_{r^2/4t}^\infty \left[z^{\nu-2}\Phi(\nu-1,\nu,-z) - \frac{z^{\nu-1}}{\nu}\Phi(\nu,\nu+1,-z)\right]dz. \tag{2.14}$$

Here $\Phi(a,c,z) = {}_1F_1(a,c,z)$ is the confluent hypergeometric function. It can be shown that this implies

$$\psi_0(r,t) = \begin{cases} \dfrac{r}{2(\nu-1)} - \dfrac{r}{2(\nu-1)}\left(\dfrac{r^2}{4t}\right)^{\nu-1}\dfrac{1}{\Gamma(\nu)}\left[1+O\left(\dfrac{r^2}{4t}\right)\right], & \text{for } \nu \neq 1, \\ \dfrac{r}{2}\left[\ln r + O(1)\right], & \text{for } \nu = 1. \end{cases} \tag{2.15}$$

But this means that for any $\nu > 0$ there exists a μ such that $0 < \mu < 1$ and

$$\left|\int_\alpha^\beta d\tau \int_\gamma^\delta \frac{\partial E}{\partial r} r^\mu \rho \, d\rho\right| < H, \qquad 0 \leq r < \infty, \quad 0 < t_0 \leq t < T, \tag{2.16}$$

whatever the nonnegative $\alpha, \beta, \gamma,$ and δ may be. Here $H > 0$ is a certain constant.

Since $F(r,t)$, as a function of bounded variation, can be represented as a sum of monotone functions, it follows from this and from the second mean-value theorem that the integral

$$i_\mu = r^\mu \int_0^t d\tau \int_0^\infty F(\rho,\tau) \frac{\partial}{\partial r} E(r,\rho,t-\tau)\rho \, d\rho \tag{2.17}$$

is uniformly bounded for $0 \leq r \leq R,\ 0 < t_0 \leq t \leq T$. Here $R > 0,\ t_0 > 0,\ T > t_0$ are arbitrary fixed numbers. But this means that

$$\lim_{\rho\to 0} \rho \frac{\partial U}{\partial \rho} E(r,\rho,t-\tau) = \lim_{\rho \to 0} \rho^{1-\mu} i_\mu(\rho,t) E(r,\rho,t-\tau) = 0,$$

which was required to be proved.

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