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Abstract

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ON BINARY HYPOTHESES IN THE THEORY OF PRIME NUMBERS BY THE METHOD OF I. M. VINOGRADOV

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Let, for a given integer $k \geqslant 1$, $\pi_k(x)$ denote the number of pairs of primes $p, p+2k$ (k-twins) lying in the interval $(0,x)$. With regard to the number $\pi_k(x)$, Hardy and Littlewood in 1923 ([2]) stated the following proposition:

$$\pi_k(x) \sim 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{x}{\ln^2 x}\prod_{\substack{p|k\\ p>2}}\left(1+\frac{1}{p-2}\right),$$

where p runs over the prime numbers.

This hypothesis is one of the most interesting and difficult in the theory of prime numbers. At present, concerning $\pi_k(x)$, only results of the type of upper estimates are known, obtained on the basis of the "sieve" method.

In the present communication we give the results of applying I. M. Vinogradov' s method of trigonometric sums ([1]) to the indicated problem and its generalizations; from these results, in particular, it follows that the Hardy-Littlewood hypothesis stated above is true almost always. The same is also valid for the hypothesis on the number $J(N)$ of representations of even N as a sum of two prime numbers.

More precise formulations of the results are as follows.

Theorem 1. *The number $\pi_k(x)$ of pairs of primes $p, p+2k$ from the interval $(0,x)$ is expressed by the asymptotic equality*

$$\pi_k(x) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{x}{\ln^2 x}\prod_{\substack{p|k\\ p>2}}\left(1+\frac{1}{p-2}\right)+O\left(\frac{x}{\ln^3 x}\right)$$

for every $0<2k\leqslant x/\ln x$, with the exception of no more than $cx/(\ln x)^M$ of them, where $M>1$ is arbitrary fixed, and the constants in the symbol O and c depend only on M.

This theorem admits a substantial generalization to progressions with increasing difference.

Theorem 2. *For every* $0 < 2k \leqslant x/\ln x$, $2k \equiv l' - l'' \pmod{D}$, *with the exception of no more than* $cx/D(\ln x)^M$ *of them, where* $M > 1$ *is any given number,* $\pi_k(x; D)$*—the number of pairs of* k*-twin primes* $p, p+2k$ *of the interval* $(0, x)$, *belonging respectively to the arithmetic progressions* $Dm + l'$, $Dn + l''$ *with the condition*

$$1 \leqslant l', l'' \leqslant D; \qquad (l', D) = (l'', D) = 1; \qquad 0 < D \leqslant (\ln x)^A,$$

where $A > 0$ *is any constant, is expressed in the form*

$$\pi_k(x; D) = 2\prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{\substack{p|D \\ p>2}} \frac{p-1}{p-2} \frac{x}{\varphi(D)\ln^2 x} \prod_{\substack{p>2 \\ p|k,\ p\nmid D}} \frac{p-1}{p-2} + O\left(\frac{x \ln\ln D}{\varphi(D)\ln^3 x}\right),$$

where p *runs over the prime numbers;* φ *is Euler' s function, and the constants in the symbol* O *and* c *depend only on* A *and* M.

Thus, the prime numbers that are k-twins, for almost all k, are asymptotically uniformly distributed over all primitive arithmetic progressions whose differences do not exceed any given power of the logarithmic length of the interval $(0, x)$.

With respect to the number of representations of even numbers as a sum of two primes, analogous theorems are obtained.

Theorem 3. *For every even* $N \equiv l' + l'' \pmod{D}$ *from the interval* $(0, x)$, *with the exception of no more than* $cx/D(\ln x)^M$ *of them, where* $M > 0$ *is arbitrary but fixed, the number* $J(N; D)$ *of solutions of the equation*

$$N = p' + p''$$

in prime numbers $p' \equiv l' \pmod{D}$, $p'' \equiv l'' \pmod{D}$, *where* $1 \leq l', l'' \leq D$, $(l', D) = (l'', D) = 1$, $0 < D \leq (\ln x)^A$, $A > 0$ *is assigned arbitrarily, is expressed by the formula*

$$J(N; D) = 2\prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{\substack{p/D \\ p>2}} \frac{p-1}{p-2} \frac{N}{\varphi(D)\ln^2 N} \prod_{\substack{p>2 \\ p/N,\ p\nmid D}} \frac{p-1}{p-2} + O\left(\frac{N \ln\ln D}{\varphi(D)\ln^3 N}\right),$$

where p *runs over prime numbers,* φ *is Euler' s function, and the constants in the symbol* O *and* c *depend only on* A *and* M.

An immediate consequence of this theorem is the validity of the Hardy–Littlewood hypothesis ([2]) concerning the order of growth of $J(N)$ for almost all even numbers N.

The theorems are derived by the method of trigonometric sums of I. M. Vinogradov ([1]), in the form of the work of N. G. Chudakov ([3]). We shall confine ourselves here to indicating its main points. Retaining the notation of Theorem 2, put, in addition:

1°. $d = (D, q)$, $g \bmod q$ with the condition $gq/d \equiv 1 \pmod{d}$, and $N \bmod q$; $N = gq/d$; $R(q) = 1$, if $\left(\frac{q}{d}, D\right) = 1$, and $R(q) = 0$ otherwise.

$$2^\circ. \quad F'_{aq}(\alpha) = R(q)\frac{\mu\left(\frac{q}{d}\right)}{\varphi\left(\frac{q}{d}\right)} e^{2\pi i \frac{a}{q} N l'} \sum_{3 \le n' \le x} \frac{e^{2\pi i n'\left(\alpha - \frac{a}{q}\right)}}{\ln n'},$$

$$F''_{aq}(\alpha) = R(q)\frac{\mu\left(\frac{q}{d}\right)}{\varphi\left(\frac{q}{d}\right)} e^{-2\pi i \frac{a}{q} N l''} \sum_{3 \le n'' \le x} \frac{e^{-2\pi i n''\left(\alpha - \frac{a}{q}\right)}}{\ln n''},$$

where μ is the Möbius function.

Taking now $\Delta = (\ln x)^\theta$, $\theta = 24(M + A + Z)$, define the function $\Phi_k(x; D)$ by the relation

$$\sum_{1 \le q \le \Delta} \sum_{\substack{1 \le a < q \\ (a,q)=1}} F'_{aq}(\alpha) F''_{aq}(\alpha) = \sum_{|k| \le x-3} \Phi_k(x; D) e^{2\pi i \alpha k}.$$

In this case, for

$$k \equiv l' - l'' \pmod{D}; \qquad 0 < k \le x/\ln x; \qquad \tau(k) \le (\ln x)^M$$

($\tau(k)$ is the number of divisors of k), $\Phi_k(x; D)$ is expressed in the form

$$\lambda \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{\substack{p/D \\ p>2}} \frac{(p-1)^2}{p(p-2)} \frac{Dx}{\ln^2 x} \prod_{\substack{p>2 \\ p/k,\ p\nmid D}} \frac{p-1}{p-2} + O\left(\frac{Dx}{\ln^3 x}\right).$$

where $\lambda = 1$, if D is an even number, and $\lambda = 2$ for odd D. On the other hand, following mainly the work of N. G. Chudakov [3], we obtain

$$\sum_{|k| < x-3} \left|\varphi^2(D)\pi_k(x; D) - \Phi_k(x; D)\right|^2 \ll x^3 \left(D (\ln x)^{M+6}\right)^{-1}, \qquad (1)$$

where the constant in the symbol $\ll$ depends only on A and M. Denoting further by $Z(x/\ln x)$ the number of integers $0 < k \leq x/\ln x,\ k \equiv l' - l'' \pmod D$, for which

$$\left|\varphi^2(D)\pi_k(x;D) - \Phi_k(x;D)\right| > Dx(\ln^3 x)^{-1}, \tag{2}$$

as a consequence of the preceding estimate we find

$$Z(x/\ln x) \ll x\big(D(\ln x)^M\big)^{-1}.$$

To these must also be added those of the indicated numbers k which satisfy the inequality:

$$\tau(k) > (\ln x)^M,$$

but the number of these latter is no more than $\ll x\big(D(\ln x)^M\big)^{-1}$, since the total number of exceptional k lying in the interval $(2, x/\ln x)$ turns out to be no more than $cx/D(\ln x)^M$, where c depends only on A and M.

Consequently, for each non-exceptional $0 < k \leq x/\ln x,\ k \equiv l' - l'' \pmod D$, the inequality opposite to inequality (1) is valid. Hence, by virtue of the asymptotic behavior of the function $\Phi_k(x;D)$, Theorem 2 follows.

In conclusion, let us note that, first, combining the indicated results of I. M. Vinogradov' s method with the results of the sieve method makes it possible to establish certain new theorems concerning the difference between neighboring pairs $[p_{kj}; p_{kj} + 2k]$, $[p_{k(j-1)}; p_{k(j-1)} + 2k]$ of prime k-twins, analogous to those known for ordinary neighboring prime numbers. Secondly, by I. M. Vinogradov' s method one may also consider the problem of the number of primes p from the interval $(0, x)$ such that all the numbers $p + u_1, \dots, p + u_m$ are prime, a problem which had been the object of investigation only by the sieve method.

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CITED LITERATURE

[1] I. M. Vinogradov, *Selected Works*, Publishing House of the Academy of Sciences of the USSR, 1952.
[2] G. Hardy, J. Littlewood, *Acta Math.*, **44**, 1 (1923).
[3] N. G. Chudakov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, No. 1, 25 (1938).

Note: Figure translations are in progress. See original paper for figures.