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Abstract

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On the Riemann Problem on Closed Riemann Surfaces*

(Presented by Academician I. N. Vekua, 18 II 1960)

Let R be a closed Riemann surface of genus p; let Γ be a contour bounding a connected domain T^+, and let T^- be its complement on R. In the note ([3]) we studied the solvability conditions for the Riemann boundary-value problem

$$\Phi^+(t) = G(t)\Phi^-(t). \tag{1}$$

Using the Riemann–Roch theorem, one can prove the following fact.

Theorem 1. The problem (1) of index $\varkappa$ has $\varkappa - p + 1$ solutions if $\varkappa > 2p - 2$; the number of solutions may vary from $\varkappa - p + 1$ to $\varkappa + 1$ for $p \leq \varkappa \leq 2p - 2$; from 0 to $\varkappa + 1$ for $0 \leq \varkappa \leq p - 1$. For $\varkappa < 0$ the problem is unsolvable.

Conjugate to (1) we shall call the following boundary-value problem: to find linear differentials $\psi^\pm(z)$, analytic respectively in the domains $T^\pm$ and satisfying the boundary condition

$$\psi^+(t) = \frac{1}{G(t)}\psi^-(t). \tag{2}$$

By division by a differential of the first kind dZ, we reduce problem (2) to the ordinary boundary-value problem of index $2p - \varkappa - 2$. With the aid of the Riemann–Roch theorem one proves

Theorem 2. The difference between the number of solutions k and $\tilde{k}$ of the conjugate boundary-value problems (1) and (2) is equal to $\varkappa - p + 1$.

Indeed, it is not difficult to see that if the family of solutions of problem (1) is determined by the divisor class (D), then the family of solutions of problem (2) corresponds to the class (W/D), whence the assertion of the theorem follows.

In the work ([3]) the nonhomogeneous boundary-value problem was also considered

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \tag{3}$$

and certain solvability conditions were indicated.

Theorem 3. For the solvability of problem (3) it is necessary and sufficient that

$$\int_\Gamma g(t)\psi_k(t) = 0 \quad (k = 1, 2, \dots, \tilde{k}), \tag{4}$$

where ψ_k is a complete system of solutions of problem (2).

The necessity of the condition is obvious. Sufficiency follows from the solvability conditions for the problem of zero index and Theorem 3 of the work ([3]).

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* The principal results of the present note were reported by the author at the Fourth All-Union Conference on the Theory of Functions of a Complex Variable in Moscow in May 1958.

This follows from consideration of the equation

$$\frac{1 + G(t)}{2}\varphi(t) + \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\left[A^+(\tau,t) - G(t)A^-(\tau,t)\right] d\tau = g(t), \tag{5}$$

to which problem (3) ([3]) and its adjoint are reduced, if one takes into account that the solutions of the adjoint equation will be the differentials $\varphi_k(t)$ $(k = 1, 2, \dots, \tilde{k})$, and only these.

Let T be a finite Riemann surface of genus h with $m + 1$ contours. Then, as shown in ([3]), the Hilbert problem

$$\mathrm{Re}[(a - ib)F(t)] = c(t) \tag{6}$$

is reduced to the Riemann problem

$$\Phi^+(t) = -\frac{a(t) + ib(t)}{a(t) - ib(t)}\Phi^-(t) + \frac{c(t)}{a(t) - ib(t)}. \tag{7}$$

on the double M of the surface T.

We shall call the problem

$$\mathrm{Re}[(a + ib)\psi(t)] = 0, \tag{8}$$

where $\psi(t)$ is a linear differential, adjoint to (6).

Theorem 4. *The number of solutions of the Hilbert problem (6) coincides with the number of solutions of the Riemann problem (7). The difference between* *the numbers of solutions of the homogeneous Hilbert problem and of its adjoint is equal to* $2\varkappa - 2h - m + 1$, *where* $\varkappa = \operatorname{ind}_\Gamma(a + ib)$. *For solvability of the nonhomogeneous Hilbert problem (7) it is necessary and sufficient that*

$$\int_\Gamma \frac{c(t)}{a(t) - ib(t)} \psi_k(t) = 0 \quad (k = 1, \dots, \tilde{l}), \tag{9}$$

where $\psi_k(t)$ $(k = 1, 2, \dots, \tilde{l})$ *is a complete system of solutions of problem (8).*

In the case $h = 0$ our results coincide with the known results of I. N. Vekua.

Let us consider the Riemann problem in a more general setting. Let Δ be a given divisor, $\operatorname{ord}\Delta = n$. Find differentials $\omega_\nu^\pm(z)$ of dimension ν, analytic respectively in the domains $T^\pm$, satisfying the boundary condition

$$\omega_\nu^+(t) = G(t)\omega_\nu^-(t), \quad (\omega_\nu) + \Delta \geqslant 0. \tag{10}$$

Dividing the boundary condition by a differential dZ^ν of dimension ν, we obtain a problem for functions

$$\Phi^+(t) = G(t)\Phi^+(t), \quad (\Phi) + \Delta + (dZ^\nu) \geqslant 0 \tag{11}$$

of index $\varkappa$, obviously equivalent to some problem

$$\Phi_1^+(t) = G_1(t)\Phi_1^-(t), \quad (\Phi) \geqslant 0 \tag{12}$$

of index $\varkappa_1 = \varkappa + 2\nu(p - 1) + \operatorname{ord}\Delta$.

The number of solutions of this problem is computed with the aid of Theorem 1. We shall call adjoint to (10) the problem

$$\eta_{-\nu+1}^+(t) = \frac{1}{G(t)}\eta_{-\nu+1}^-(t), \quad (\eta_{-\nu+1}) - \Delta \geqslant 0. \tag{13}$$

Dividing the boundary condition by the differential $dZ^{-\nu} = (dZ^\nu)^{-1}$ of dimension $-\nu$, we can reduce it to the form

$$\psi^+(t)_1 = \frac{1}{G_1(t)}\psi^-(t), \qquad (\psi) \geqslant 0 \tag{14}$$

for linear differentials.

Problems (12) and (14) are, obviously, adjoint.

From Theorem 2 it follows

Theorem 5. *The difference between the numbers of solutions of the homogeneous adjoint problems (10) and (13) is equal to*

$$\varkappa + (2\nu - 1)(p - 1) + \operatorname{ord} \Delta,$$

where $\varkappa = \operatorname{ind}_\Gamma G$.

With the aid of Theorem 3 one proves

Theorem 6. *For the solvability of the nonhomogeneous problem*

$$\omega_\nu^+(t) = G(t)\omega_\nu^-(t) + g_\nu(t), \qquad (\omega_\nu) + \Delta \geqslant 0 \tag{15}$$

it is necessary and sufficient that

$$\int_\Gamma g_\nu(t)\eta^+_{-\nu+1}(t) = 0, \tag{16}$$

where $\eta^+_{-\nu+1}(t)$ *is an arbitrary solution of the homogeneous problem (13).*

Using the method of paper ([3]), one can, with the help of these results, investigate the Riemann-Hilbert problem. In doing so, obviously, all the results of Koppelman' s paper ([4]) follow from our results.

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[1] I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1958. [2] M. Schiffer, D. C. Spencer, *Functionals on Finite Riemann Surfaces.* Moscow, 1957. [3] Yu. L. Rodin, DAN, **129**, No. 6 (1959). [4] W. Koppelman, Comm. Pure and Appl. Math., **12**, 13 (1959).

Note: Figure translations are in progress. See original paper for figures.