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Abstract

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CONFORMAL THEORY OF TWO-PARAMETER FAMILIES OF SPHERES

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1°. Three-dimensional conformal space C_3 is the three-dimensional Euclidean space completed by a point at infinity, whose fundamental group is the 10-parameter group of point transformations carrying spheres into spheres. As is known, these transformations exhaust all conformal transformations of three-dimensional space.

Consider in C_3 a family of spheres (S) depending on two parameters u^1 and u^2. We refer this family to a local conformal Cartan frame ([1], p. 171), consisting of three mutually orthogonal spheres S_1, S_2, and S_3, and two points S_0 and S_4 through which they pass; moreover, as the sphere S_1 we take the sphere S of the family. The equations of the infinitesimal displacement of this frame have the form

$$dS_\alpha = \omega_\alpha^\beta S_\beta \qquad (\alpha, \beta, \gamma = 0, 1, 2, 3, 4), \tag{1}$$

where the Pfaff forms ω_α^β satisfy the structure equations—the conditions of complete integrability of system (1), which in the symbolism of exterior forms have the form ([2])

$$D\omega_\alpha^\beta = [\omega_\alpha^\gamma \omega_\gamma^\beta]. \tag{2}$$

In addition, the 25 forms ω_α^β are connected by 15 linear relations, which for the Cartan frame take the form

$$\omega_j^i + \omega_i^j = 0; \qquad \omega_i^4 + \omega_0^i = 0; \qquad \omega_4^i + \omega_i^0 = 0;$$

$$\omega_4^0 = 0, \qquad \omega_0^4 = 0; \qquad \omega_4^4 + \omega_0^0 = 0 \qquad (i, j = 1, 2, 3). \tag{3}$$

2°. The differential equations of the two-parameter family of spheres (S_1) under consideration have the form ([3])

$$\omega_1^p = \lambda_{1\chi}^p du^\chi \qquad (p, q = 0, 2, 3, 4;\ \chi, \lambda, \mu, \nu = 1, 2). \tag{4}$$

Continuing this system, we obtain:

$$d\lambda_{1\chi}^p = -\lambda_{1\chi}^q \omega_q^p + \lambda_{1\chi\lambda}^p du^\lambda. \tag{5}$$

Consequently, the quantities $g_{\alpha\beta}$ (the coordinates of the tensor of the angular metric in C_3, equal for the Cartan frame to 0 and 1) and $\lambda_{1\chi}^p$ are, for the family of spheres (S_1), components of a fundamental geometric object of first order.

Continuing system (4) once more, we obtain

$$d\lambda_{1\chi\lambda}^p = -\lambda_{1\chi\lambda}^q \omega_q^p + (\ \ )_\mu du^\mu + \lambda_{1\chi\lambda\mu}^p du^\mu. \tag{6}$$

Thus, the quantities $g_{\alpha\beta}$, $\lambda_{1\chi}^p$, $\lambda_{1\chi\lambda}^p$ form the fundamental object of the family (S_1) of the second order, while the totality of the quantities $g_{\alpha\beta}$, $\lambda_{1\chi}^p$, $\lambda_{1\chi\lambda}^p$, $\lambda_{1\chi\lambda\mu}^p$ is the fundamental geometric object of the third order. The following basic theorem holds:

Theorem 1. *The fundamental geometric object of the third order is complete: specifying the field of the fundamental object of the third order (i.e., the values of its components at each point of the domain under consideration in parameter space) determines the family of spheres (S_1) up to a conformal transformation.*

3°. The fundamental object of the first order includes the absolute tensor $a_{\chi\lambda}$. This tensor generates the absolutely invariant differential form

$$\Phi = a_{\chi\lambda}\, du^\chi du^\lambda. \tag{7}$$

This form gives the **angular linear element of the family of spheres**–the square of the angle between infinitely close spheres S_1 and $S_1 + dS_1$ of the family.

The null lines of this form single out two "directions" $du^2 : du^1$ in which the infinitely close spheres S_1 and $S_1 + dS_1$ of the family are tangent, i.e., generate a parabolic pencil π. In each of these parabolic pencils there is a unique sphere S whose derivative under displacement in the other "direction" belongs to the same parabolic pencil π. We shall call these two "directions," which make Φ vanish, **conjugate**. Thus, the given two-parameter family of spheres (S_1), along each of the conjugate "directions," is transformed into a new family of spheres (S) with the same conjugate net of "directions."The same transformation can again be applied to this family (S), yielding a sequence of transformations of two-parameter families of spheres. Note that these conjugate directions are imaginary for families having real envelopes, and real for two-parameter families having imaginary envelopes.

In the general case the tensor $a_{\chi\lambda}$ is nondegenerate, and there exists the inverse tensor $a^{\chi\lambda}$.

The fundamental object of the second order includes the four-valent covariant absolute tensor $\overset{2}{a}_{\chi\lambda\mu\nu}$. The differential form

$$\overset{2}{\Phi} = \overset{2}{a}_{\chi\lambda\mu\nu}\, du^\chi du^\lambda du^\mu du^\nu, \tag{8}$$

generated by this tensor, gives the square of the stationary angle of the sphere $S_1 + dS_1 + \frac{1}{2} d^2 S_1$ of second differential osculation with the spheres of the derived bundle dS_1. Contracting the tensor $\overset{2}{a}_{\chi\lambda\mu\nu}$ with the tensor $a^{\chi\lambda}$, we obtain the absolute tensor $\overset{2}{a}_{\mu\nu}$

$$\overset{2}{a}_{\mu\nu} = \overset{2}{a}_{\chi\lambda\mu\nu} a^{\chi\lambda}. \tag{9}$$

Now we obtain the absolute invariants of the family of spheres (S_1):

$$\overset{2}{I}_1 = \frac{\operatorname{Det}\left|\overset{2}{a}_{\chi\lambda}\right|}{\operatorname{Det}\left|a_{\chi\lambda}\right|}; \qquad \overset{2}{I}_2 = \overset{2}{a}_{\mu\nu} a^{\mu\nu}. \tag{10}$$

In addition, the fundamental object of the second order includes the relative four-valent covariant tensors $b_{\chi\lambda}$, $c_{\chi\lambda}$, $e_{\chi\lambda}$. These

the tensors generate relatively invariant quadratic differential forms:

$$\psi_1 = b_{\chi\lambda} du^\chi du^\lambda; \qquad \varphi_2 = c_{\chi\lambda} du^\chi du^\lambda; \qquad \varphi_3 = e_{\chi\lambda} du^\chi du^\lambda. \tag{11}$$

The zero lines of these forms will be invariant. The equation $\varphi_1 = 0$ defines the net of curvature lines of one sheet of the envelope of the family of spheres (S_1); the equation $\varphi_2 = 0$, the net of curvature lines of the second sheet of the envelope. The equation $\varphi_3 = 0$ singles out two directions such that the characteristic circles $[S_1, dS_1]$ along one direction, when the sphere is displaced in the other direction, are cospherical—lie on one sphere T. Thus the two-parameter family of spheres (S_1), along each of the two directions determined by the equality $\varphi_3 = 0$, is transformed into a family of spheres (T) with the same form φ_3. This family of spheres (T) may again be transformed along the same direction, and in this way one obtains a sequence of transformations of the given two-parameter family of spheres (S_1) along the directions determined by the equality $\varphi_3 = 0$. We shall call these directions the **principal** ones.

4°. Suppose that the family of spheres (S_1) has two distinct sheets of the envelope. We then place the points S_0 and S_4 of the frame at the characteristic points of the sphere S_1. Then $(dS_1 S_0) = 0$ and $(dS_1 S_4) = 0$, whence we obtain

$$\omega_0^1 = 0; \qquad \omega_1^0 = 0. \tag{12}$$

Differentiating these equations exteriorly and expanding the resulting quadratic equations by Cartan' s lemma, we obtain ([2])

$$\omega_0^2 = \alpha\omega_1^2 + \beta\omega_1^3; \qquad \omega_0^3 = \beta\omega_1^2 + \gamma\omega_1^3; \qquad \omega_2^0 = \alpha_1\omega_1^2 + \beta_1\omega_1^3; \qquad \omega_3^0 = \beta_1\omega_1^2 + \gamma_1\omega_1^3. \tag{13}$$

Under this specialization the indicated invariant differential forms have the form

$$\Phi = (\omega_1^2)^2 + (\omega_1^3)^2; \qquad \overset{2}{\Phi} = -\frac{1}{2}(\omega_1^2\omega_2^0 + \omega_1^3\omega_3^0)(\omega_1^3\omega_3^0 + \omega_1^2\omega_2^0);$$

$$\varphi_1 = \omega_1^2\omega_0^3 - \omega_1^3\omega_0^2 = \beta(\omega_1^2)^2 + (\gamma - \alpha)\omega_1^2\omega_1^3 - \beta(\omega_1^3)^2,$$

$$\varphi_2 = \omega_1^2\omega_3^0 - \omega_1^3\omega_2^0 = \beta_1(\omega_1^2)^2 + (\gamma_1 - \alpha_1)\omega_1^2\omega_1^3 - \beta_1(\omega_1^3)^2,$$

$$\varphi_3 = \omega_2^0\omega_0^3 - \omega_3^0\omega_0^2 = (\alpha_1\beta - \alpha\beta_1)(\omega_1^2)^2 + (\alpha_1\gamma - \alpha\gamma_1)\omega_1^2\omega_1^3 + (\beta_1\gamma - \beta\gamma_1)(\omega_1^3)^2. \tag{14}$$

As early as the end of the nineteenth century, Ribaucour and Darboux investigated two-parameter families of spheres for which the nets of curvature lines corresponded on both sheets of the envelope. These families were called R-systems. They are characterized by the proportionality of the forms φ_1 and φ_2. But then, as follows from equalities (14), the form φ_3 will also be proportional to them, and the rank of the system of forms φ_1, φ_2, and φ_3 for R-systems is equal to one. Conversely, families of spheres for which the rank of the system of forms φ_1, φ_2, and φ_3 is equal to one are R-systems.

Consider the class of families of spheres for which the form $\overset{2}{\Phi}$ is the square of a quadratic form. This can occur in two cases: a) each of the factors of the form $\overset{2}{\Phi}$ is the square of a linear form; then one can show that the envelopes (S_0) and (S_4) of the family of spheres (S_1) degenerate into lines; b) both factors of the form $\overset{2}{\Phi}$ are proportional.

This class of families of spheres in a canonized frame is determined by the conditions

$$\beta = 0; \qquad \beta_1 = 0; \qquad \alpha_1 = t\alpha; \qquad \gamma_1 = t\gamma. \tag{15}$$

Hence it is seen that this is a special class of systems R. The system of Pfaff equations determining this class of families of spheres has, according to equalities (12), (13), and (15), the form

$$\omega_0^1 = 0; \quad \omega_1^0 = 0; \quad \omega_2^0 = \alpha\omega_1^2; \quad \omega_0^3 = \gamma\omega_1^3; \quad \omega_2^0 = t\omega_0^2; \quad \omega_3^0 = t\omega_0^3. \tag{16}$$

The prolongation of this system has the form

$$d\alpha - \alpha\omega_0^0 = a\omega_1^2 + b\omega_1^3; \quad (\alpha - \gamma)\omega_2^3 = b\omega_1^2 + c\omega_1^3;$$

$$d\gamma - \gamma\omega_0^0 = c\omega_1^2 + e\omega_1^3; \quad dt + 2t\omega_0^0 = 0. \tag{17}$$

The last equation of system (17) is completely integrable. This class of families of spheres exists with one arbitrary function of two arguments.

Theorem 2. *The family of spheres (16) and (17) belongs to a net of spheres.*

Indeed, by virtue of equalities (17), (16), the sphere $S = tS_0 + S_4$ remains constant for the family of spheres (S_1). Consequently, all spheres of the family (S_1) belong to a net—to the linear family generated by the four spheres $S_1, S_2, S_3, tS_0 - S_4$. The congruence of orthogonal circles to the family (16) is orthogonal to the sphere $tS_0 + S_4$, and, consequently, in this congruence the two pairs of focal surfaces of the net of curvature lines correspond to each other, and both these pairs are in conformal correspondence ((4), p. 139).

5°. A two-parameter family of spheres $(\bar{S}_1)$, conformally superposable in the sense of Cartan on a family (S_1) with order of superposition two (5), shall be called a **bending of the family** (S_1).

Theorem 3. *The only class of two-parameter families of spheres admitting a conformal bending is the class for which the invariant form* $\overset{2}{\Phi}$ *is the fourth power of a linear form* ω*, and the one-parameter subfamilies of spheres singled out by the condition that the form* ω *vanish are pencils.*

This class of families exists with the arbitrariness of three functions of one argument, and each family of this class admits a conformal bending with the arbitrariness of one function of one argument. Both sheets of the envelope for this class of families are expressed as lines, and the congruence of orthogonal circles is a special class of Ribaucour cyclic systems, for which one pair of focal surfaces is expressed as two curves coinciding with the enveloping families of spheres, while the other two represent one and the same canal surface.

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Note: Figure translations are in progress. See original paper for figures.