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Abstract

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REMARKS ON THE SOLUTION OF MULTIDIMENSIONAL SINGULAR INTEGRAL EQUATIONS

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1°. In the note ([1]), theorems were formulated asserting normal solvability and equality to zero of the index in the space $\mathcal{L}_p(E_m)$ for a multidimensional singular equation with a symbol satisfying certain smoothness conditions and nowhere vanishing.* On the coefficient $a(x)$ and the characteristic $f(x,\theta)$ of the singular equation

$$a(x)u(x) + \int_{E_m} \frac{f(x,\theta)}{r^m} u(y)\, dy + Tu = g(x) \tag{1}$$

(T is an operator completely continuous in $\mathcal{L}_p(E_m)$) rather complicated additional conditions were imposed (inequalities (16) of note ([1])). Their meaning in application, for example, to the characteristic $f(x,\theta)$ is that, if E_m is mapped stereographically onto the unit sphere Σ of the space E_{m+1}, and to the points $x, x', y \in E_m$ there correspond the points $\xi, \xi', \eta \in \Sigma$, then the inequality

$$|f(x,\theta) - f(x',\theta)| \le A|\xi - \xi'|^\gamma, \tag{2}$$

must be satisfied, where the constants A and γ do not depend on θ.

In the present note we shall indicate a condition, simpler and more general than condition (2), and sufficient for obtaining the above-mentioned results of ([1]). We shall also indicate a new device for reducing equation (1) to an equation containing a completely continuous operator—a device less general, but simpler, than in article ([2]).

2°. Theorem 1. *Let the symbol $\Phi(x,\theta)$ of some singular operator be measurable and bounded with respect to the aggregate of points x and θ, and have all generalized derivatives of order $m-1$ with respect to the angular coordinates $\vartheta_1, \vartheta_2, \dots, \vartheta_{m-1}$ of the point θ, summable with degree p' over the parallelepiped Π, defined by the inequalities $0 \le \vartheta_j \le \pi,\ j < m-1$, and $-\pi \le \vartheta_{m-1} \le \pi$, the corresponding integrals being bounded independently of x. Then the singular operator is bounded in $\mathcal{L}_p(E_m)$.*

This theorem is proved in the same way as Theorem 1 of note ([1]).

We shall say that a certain function $\varphi(x,\theta)$ belongs to the space $W_p^{(l)}(S)$, where S is the unit sphere of the space E_m, *uniformly with respect to* x, and write $\varphi(x,\theta) \in W_p^{(l)}(S)$, if $\varphi(x,\theta) \in W_p^{(l)}(S)$ for every fixed x and the integrals

$$\int_S |\varphi(x,\theta)|^p \, dS; \qquad \int_S |D^l \varphi(x,\theta)|^p \, dS$$

* We use here the notation and terminology of papers ([1,2]).

are bounded independently of x; D^l denotes any derivative of order l with respect to the Cartesian coordinates of the point θ. For $p \geqslant 2$ the conditions of Theorem 1 are obviously satisfied if $\Phi(x,\theta) \hat{\in} W_2^{(m)}(S)$; as can be derived from the results of the paper ([3]), for $p < 2$ the conditions of Theorem 1 are satisfied if $\Phi(x,\theta) \hat{\in} W_2^{(l)}(S)$, $l \geqslant (m-1)\left(\frac{3}{2} - \frac{1}{p'}\right) + 1$.

Theorem 2. Let the symbol $\Phi(x,\theta)$ of the singular operator A be expanded in a series in spherical functions

$$\Phi(x,\theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} a_n^{(k)}(x) Y_{n,m}^{(k)}(\theta). \tag{3}$$

If $\Phi(x,\theta) \hat{\in} W_2^{(l)}(S)$, where $l = m$ for $p \geqslant 2$ and $l \geqslant (m-1)\left(\frac{3}{2} - \frac{1}{p}\right) + 1$ for $p < 2$, then the operator A is expanded in a norm-convergent series

$$A = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} a_n^{(k)}(x) A_n^{(k)} + T, \tag{4}$$

where

$$A_0^{(1)} = I, \qquad A_n^{(k)} u = \frac{\Gamma((n+m)/2)}{\pi^{m/2} i^n \Gamma(n/2)} \int_{E_m} \frac{Y_{n,m}^{(k)}(\theta)}{r^m} \, u(y) \, dy$$

is a singular operator with symbol $Y_{n,m}^k(\theta)$; T is a certain completely continuous operator in $\mathcal{L}_p(E_m)$.

The proof is based on Theorem 1 and on the fact that, by the results of the paper ([3]), the remainder of the series (3) tends in the metric $W_p^{(m-1)}(S)$ to zero uniformly with respect to x.

Lemma. Let D be a finite domain of the space E_m, containing the origin either inside or on the boundary; let λ be a constant lying in the range $0 \leqslant \lambda < m$;

and let $A(x, y)$ be a bounded measurable function. The integral operator with weak singularity

$$\int_D \frac{A(x,y)}{r^\lambda}\, u(y)\, dy$$

is bounded in the space $\mathcal{L}_p(|x|^\beta; D)$ of functions summable in D to the power p and with weight $|x|^\beta$, if $-m < \beta < mp/p'$.

The proof is based on Hölder' s inequality and on the theorem on the kernel of the product of integral operators with weak singularities.

Theorem 3. Let A and B be singular operators with symbols $\Phi_A(x,\theta)$ and $\Phi_B(x,\theta)$. Suppose that $\Phi_A, \Phi_B \hat{\in} W_2^{(l)}(S)$, where $l \geqslant m$ for $p \geqslant 2$, and $l \geqslant (m-1)\left(\frac{3}{2} - \frac{1}{p'}\right) + 1$ for $p < 2$, and that the stereographic transformation takes the functions $\Phi_A(x,\theta)$ and $\Phi_B(x,\theta)$ into functions of ξ and θ, continuous on Σ with respect to ξ uniformly with respect to θ. Then the operator $AB-BA$ is completely continuous in $\mathcal{L}_p(E_m)$, and the symbol of the product AB is equal to the product $\Phi_A(x,\theta)\Phi_B(x,\theta)$.

The results listed above make it possible to establish that, for singular equations whose symbols satisfy the conditions of Theorem 3, the theorems on normal solvability and on the index being equal to zero in the space are valid.

3°. In the paper ([2]) a method was indicated for an equivalent regularization of a singular equation whose symbol nowhere vanishes and satisfies certain smoothness conditions. We indicate here another method, simpler, but requiring some additional restrictions.

Suppose that the symbol of equation (1), $\Phi(x,\theta) \in W_2^{(m)}(S)$, is continuous on Σ uniformly with respect to θ. As usual, we assume that the symbol nowhere vanishes; we also make the following additional assumption: the points $\xi = 0$ and $\xi = \infty$ of the complex λ-plane can be joined by a smooth curve L not intersecting the set of values of the symbol. Consider the equation

$$u - \lambda(u - Au) + Tu = g(x); \tag{5}$$

let in this equation λ lie on the curve $\widetilde{L}$ into which the curve L is transformed under the mapping $\xi = (\lambda-1)/\lambda$. We may assume that L does not pass through the point $\xi = 1$, and then the curve $\widetilde{L}$ is bounded; the endpoints of this curve coincide with the points $\lambda = 0$ and $\lambda = 1$.

It is not hard to see that the symbol of equation (5), equal to $1 - \lambda + \lambda\Phi$, is bounded below in modulus by a positive constant independent of λ. Hence the following follows: if $\lambda \in \widetilde{L}$, then the singular operator

$$H_\lambda u = \int_\Pi \frac{1-\Phi(x,\theta)}{1-\lambda+\lambda\Phi(x,\theta)}\, d\varepsilon(\theta) u,$$

whose symbol is equal to $(1-\Phi)/(1-\lambda+\lambda\Phi)$, is bounded in norm by some constant C, which is independent of λ.

Putting $\lambda = 0$, we find that $\|I-A\| \le C$, and therefore, if $|\lambda_0| \le 1/2C$, then the operator $[I-\lambda_0(I-A)]^{-1}$ is defined on the whole space $\mathcal{L}_p(E_m)$ and is bounded; its symbol is equal to $(1-\lambda_0+\lambda_0\Phi)^{-1}$. Multiplying both sides of equation (5) on the left by the indicated operator, we arrive at an equivalent equation with symbol

$$\frac{1-\lambda+\lambda\Phi}{1-\lambda_0+\lambda_0\Phi} = 1 - \frac{(\lambda-\lambda_0)(1-\Phi)}{1-\lambda_0+\lambda_0\Phi}.$$

We multiply both sides of the equation on the left by the operator $[I-(\lambda-\lambda_1)H_{\lambda_1}]^{-1}$, where $|\lambda_1-\lambda_0| \le 1/2C$, which in turn will lead us to an equation with symbol

$$1 - \frac{(\lambda-\lambda_1)(1-\Phi)}{1-\lambda_1+\lambda_1\Phi},$$

equivalent to equation (5). Continuing this process, in the case of interest to us $\lambda = 1$, after k steps we obtain an equation with symbol

$$1 - \frac{(1-\lambda_k)(1-\Phi)}{1-\lambda_k+\lambda_k\Phi}.$$

For k sufficiently large one can make $\lambda_k = 1$, and we arrive at an equation with symbol 1, as required; by construction it is equivalent to equation (1).

The example of a one-dimensional singular equation shows that, in the general case, the "continuation in a parameter" device indicated here is not applicable if the set of values of the symbol does not satisfy the condition formulated above. However, if $p = 2$, then this condition can be discarded, replacing it by the following: equation (1) is solvable. Indeed, then this equation is equivalent to the following:

$$(A^*+T^*)(A+T)u = (A^*+T^*)g. \tag{6}$$

The symbol of equation (6) is equal to $|\Phi(x,\theta)|^2$; as the curve L for this equation one may take the real negative half-axis, and, consequently, the method of continuation in a parameter is applicable to equation (6).

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CITED LITERATURE

[1] S. G. Mikhlin, DAN, 117, No. 1 (1957). [2] S. G. Mikhlin, Vestn. LGU, No. 1 (1956). [3] S. G. Mikhlin, DAN, 126, No. 2 (1959).

Note: Figure translations are in progress. See original paper for figures.