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Abstract

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MATHEMATICS

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ON HOMOGENEOUS DIFFERENCE SCHEMES OF HIGH ORDER OF ACCURACY

1. The present work is devoted to the construction of an exact difference scheme and of difference schemes of arbitrary order of accuracy for boundary-value problems of the 1st, 2nd, and 3rd kinds for the differential equation

$$L^{(p,q,f)}u = \frac{d}{dx}\left[\frac{1}{p(x)}\frac{du}{dx}\right] - q(x)u + f(x) = 0 \qquad (0 < x < 1), \tag{1}$$

where $0 < K_1 \leq p(x) \leq K_2, \quad 0 \leq q(x) \leq K_2$.

All the schemes indicated belong to the family of homogeneous three-point conservative difference schemes ([1,2]) of the form

$$L_h^{(p,q,f)}y_i = \frac{1}{h^2}\Delta\left(\frac{\nabla y_i}{A_i^h}\right) - D_i^h y_i + \Phi_i^h, \tag{2}$$

$$\Delta y_i = y_{i+1} - y_i; \qquad \nabla y_i = y_i - y_{i-1} \qquad (h = 1/N,\ x_i = ih);$$

$$A_i^h = A^h[p(x_i + sh),\, q(x_i + sh)] \qquad (-1 < s < 0);$$

$$D_i^h = D^h[p(x_i + sh),\, q(x_i + sh)] \qquad (-1 < s < 1);$$

$$\Phi_i^h = \Phi^h[p(x_i + sh),\, q(x_i + sh),\, f(x_i + sh)] \qquad (-1 < s < 1),$$

$A^h[\bar{p}(s),\, \bar{q}(s)],\ D^h[\bar{p}(s),\, \bar{q}(s)],\ \Phi^h = \Phi^h[\bar{p}(s),\, \bar{q}(s),\, \bar{f}(s)]$ are certain nonlinear functionals computed by means of quadrature formulas of finite order of accuracy.

2. We shall give the construction of the exact scheme. Introduce a local coordinate system by setting $s = \frac{1}{h}(x - x_i)$, $x_i = ih$. Then the interval (x_{i-1}, x_{i+1}) is transformed into the interval $-1 < s < 1$, and equation (1) takes the form

$$L^*\bar{u} = \frac{d}{ds}\left[\frac{1}{\bar{p}(s)}\frac{du}{ds}\right] - h^2\bar{q}(s)\bar{u}(s) = -h^2\bar{f}(s), \tag{3}$$

where $\bar{p}(s) = p(x_i + sh)$, etc.

To construct the exact scheme it is sufficient to establish a relation connecting the values $\bar{u}(s)$ at $s = -1, 0, 1$; this is possible, since the solution of the second-order equation on the interval is determined by prescribing the values of the sought function at the endpoints of this interval.

We represent the general solution of equation (3) in the form

$$\bar{u}(s) = \frac{\alpha(s;h)}{\alpha(1;h)}\bar{u}(1) + \frac{\beta(s;h)}{\beta(1;h)}\bar{u}(-1) + h^2\gamma(s;h), \tag{4}$$

where $\alpha(s;h) = v_1$ and $\beta(s;h) = v_2$ are two linearly independent solutions of the homogeneous equation

$$L^*v = 0, \qquad \alpha(-1,h) = 0, \qquad \alpha'(-1,h) = \bar{p}(-1),$$

$$\beta(1;h) = 0, \qquad \beta'(1;h) = -\bar{p}(1), \tag{5}$$

and $\gamma(s;h) = v_3$ is the solution of the nonhomogeneous equation

$$L^*\gamma = -\bar{f}(s), \qquad \gamma(-1;h) = \gamma(1;h) = 0. \tag{6}$$

Setting $s = 0$ in (4), we arrive at the formula

$$\bar{u}(0) = P^h\bar{u}(-1) + Q^h\bar{u}(1) + R^h, \tag{7}$$

where the coefficients

$$P^h = \frac{\beta(0;h)}{\beta(-1;h)} = P^h[\bar{p}(s);\bar{q}(s)], \qquad Q^h = \frac{\alpha(0;h)}{\alpha(1;h)} = Q^h[\bar{p}(s),\bar{q}(s)],$$

$$R^h = h^2\gamma(0;h) = R^h[\bar{p}(s),\bar{q}(s);\bar{f}(s)] \tag{8}$$

are functionals of the coefficients of equation (3). Returning to the old variables, we obtain the difference equations

$$u_i = P_i^h u_{i-1} + Q_i^h u_{i+1} + R_i^h, \qquad 0 < i < N, \qquad u_0 = \mu_1, \quad u_N = u(1) = \mu_2, \tag{9}$$

which are satisfied by the solution $u = u(x)$ of the differential equation (1). Here

$$u_i = u(x_i), \quad P_i^h = P^h[\bar{p}_i(s), \bar{q}_i(s)], \quad Q_i^h = Q^h[\bar{p}_i(s), \bar{q}_i(s)],$$

$$R_i^h = R^h[\bar{p}_i(s), \bar{q}_i(s), \bar{f}_i(s)], \quad \bar{p}_i(s) = p(x_i + sh), \quad \bar{q}_i(s) = q(x_i + sh),$$

$$\bar{f}_i(s) = f(x_i + sh).$$

The functions $\alpha(s;h)$, $\beta(s;h)$, and $\gamma(s;h)$, by means of which the functionals P^h, Q^h, R^h for the exact scheme are constructed, will henceforth be called template functions.

p. 3. Let us now consider the boundary condition of the third kind at $x = 0$:

$$u'(0)/p(0) - \sigma u(0) = \mu_1 \tag{10}$$

and find its difference equivalent.

Considering the interval $(0; h)$, and then representing the general solution of equation (3) in the interval $(0 < s < 1)$ in terms of the template functions $\alpha^*(s;h)$, $\beta(s;h)$, $\gamma^*(s;h)$, satisfying the conditions

$$L^*\alpha^* = L^*\beta = 0, \qquad L^*\gamma^* = -\bar{f}(s), \qquad \alpha^*(0;h) = \gamma^*(0;h) = 0,$$

$$\beta(1;h) = \gamma^*(1;h) = 0, \qquad \alpha^{*\prime}(0;h) = \bar{p}(0), \qquad \beta'(1;h) = -\bar{p}(1), \tag{11}$$

and requiring that condition (10) be fulfilled, as a result we arrive at the exact two-point difference boundary condition of the third kind

$$u_0 = a_1 u_1 + b_1, \tag{12}$$

$$a_1 = \left\{1 + h\left[\sigma + h\int_0^1 q(sh)\overset{0}{\beta}(s;h)\,ds\right]\right\}^{-1}, \qquad b_1 = h\left[\mu_1 - \frac{h\overset{0}{\gamma}{}^{*\prime}(0;h)}{p(0)}\right]a_1\overset{0}{\beta}(0;h).$$

The superscript zero (for example, $\overset{0}{\beta}$) means that the template functions are taken for the coefficients $\bar{p}_0(s) = p(sh)$, $\bar{q}_0(s) = q(sh)$, $\bar{f}_0(s) = f(sh)$.

para. 4. Transform the difference equation (7) or (9). To this end we use a number of properties of the stencil functions $\alpha(s;h)$ and $\beta(s;h)$:

$$1) \qquad \alpha(s;h) > 0, \quad \beta(s;h) > 0 \quad \text{for } -1 < s < 1, \text{ if } \bar{q}(s) \geq 0; \tag{13}$$

$$2) \qquad \alpha(1;h) = \beta(-1;h); \tag{14}$$

$$3) \qquad \alpha(1;h) - \alpha(0;h) - \beta(0;h) = h^2 \left\{ \beta(0;h) \int_{-1}^{0} \bar{q}(s)\alpha(s;h)\,ds + \right. \\ \left. + \alpha(0;h) \int_{0}^{1} \bar{q}(s)\beta(s;h)\,ds \right\}. \tag{15}$$

Formulas (14) and (15) are established with the aid of Green' s second formula, taking into account conditions (5).

By virtue of conditions (14) and (15), equality (7) can be written as follows:

$$\frac{1}{h^2}\left[\frac{\bar{u}(1)-\bar{u}(0)}{\beta(0;h)} - \frac{\bar{u}(0)-\bar{u}(-1)}{\alpha(0;h)}\right] - \bar{u}(0)\left\{\frac{1}{\alpha(0;h)}\int_{-1}^{0}\bar{q}(s)\alpha(s;h)\,ds + \right. \\ \left. + \frac{1}{\beta(0;h)}\int_{0}^{1}\bar{q}(s)\beta(s;h)\,ds\right\} \\ + \frac{\alpha(1;h)}{\alpha(0;h)\beta(0;h)}\gamma(0;h) = 0. \tag{16}$$

Put $\bar{p}_i(s) = p(x_i + sh)$, etc., and denote by $\alpha_i(s;h)$, $\beta_i(s;h)$, $\gamma_i(s,h)$ the functions α, β, and γ obtained as a result of such a replacement. It is not difficult to show that

$$\beta_i(0;h) = \alpha_{i+1}(0;h). \tag{17}$$

As a result we obtain the conservative scheme (2), where

$$A_i^h = \alpha_i(0;h) = A^h[\bar{p}_i(s), \bar{q}_i(s)]; \quad \bar{p}_i(s) = p(x_i + sh); \quad \bar{q}_i(s) = q(x_i + sh); \tag{18}$$

$$D_i^h = \frac{1}{A_i^h}\int_{-1}^{0}\bar{q}_i(s)\alpha_i(s;h)\,ds + \frac{1}{A_{i+1}^h}\int_{0}^{1}\bar{q}_i(s)\beta_i(s;h)\,ds; \tag{19}$$

$$\Phi_i^h = \left(h^2 D_i^h + \frac{1}{A_i^h} + \frac{1}{A_{i+1}^h}\right)\gamma_i(0;h). \tag{20}$$

Condition (17) is, obviously, the condition of conservativity $B_i^h = A_{i+1}^h$ (see (2)). The equation $L_h^{(p,q,f)} y_i = 0$, determined by formulas (2), (18), (19), and (20), is, by construction, equivalent to equation (9).

Thus, it has been established that the exact scheme $L_h^{(p,q,f)}$ is a homogeneous, three-point, conservative difference scheme.

para. 5. The stencil functions $\alpha(s;h)$, $\beta(s;h)$, $\gamma(s;h)$ are entire analytic functions of h^2:

$$\alpha(s;h) = \sum_{k=0}^{\infty} h^{2k}\alpha^{(2k)}(s) = \alpha^{(0)}(s) + h^2\alpha^{(2)}(s) + \cdots + h^{2k}\alpha^{(2k)}(s) + \cdots,$$

$$\beta(s;h) = \sum_{k=0}^{\infty} h^{2k}\beta^{(2k)}(s), \qquad \gamma(s;h) = \sum_{k=0}^{\infty} h^{2k}\gamma^{(2k)}(s). \tag{21}$$

Taking into account conditions (5) and (6), we find

$$\alpha^{(0)}(s) = \int_{-1}^{s} \bar{p}(s)\,ds, \qquad \beta^{(0)}(s) = \int_{s}^{1} \bar{p}(s)\,ds, \tag{22}$$

$$\alpha^{(2k+2)}(s) = \int_{-1}^{s} \bar{p}(t)\left[\int_{-1}^{t} \bar{q}(\lambda)\alpha^{(2k)}(\lambda)\,d\lambda\right] dt,$$

$$\beta^{(2k+2)}(s) = \int_{s}^{1} \bar{p}(t)\left[\int_{t}^{1} \bar{q}(\lambda)\beta^{(2k)}(\lambda)\,d\lambda\right] dt \qquad (k = 0, 1, 2, \ldots);$$

$$\gamma^{(2k)}(s) = \left[\int_{-1}^{1} \bar{p}(t)\,dt\right]^{-1} \cdot \left\{\int_{-1}^{1} \bar{p}(t)\left[\int_{-1}^{t} \eta^{(2k)}(\lambda)\,d\lambda\right] dt \cdot \int_{-1}^{s} \bar{p}(t)\,dt \right.$$
$$\left. - \int_{-1}^{s} \bar{p}(t)\left[\int_{-1}^{t} \eta^{(2k)}(\lambda)\,d\lambda\right] dt \cdot \int_{-1}^{1} \bar{p}(t)\,dt \right\}, \tag{23}$$

where $\eta^{(0)}(\lambda) = \bar{f}(\lambda)$, $\eta^{(2k)}(\lambda) = \bar{q}(\lambda)\gamma^{(2k-2)}(\lambda)$ for $k = 1, 2, \ldots$.

Thus, each of the coefficients $\alpha^{(2k)}$, $\beta^{(2k)}$, $\gamma^{(2k)}$ is expressed, by means of $(k-1)$-fold integration, in terms of $\bar{p}(s)$, $\bar{q}(s)$, and $\bar{f}(s)$.

§ 6. The question arises: if in (21) one restricts oneself to a finite number of terms, i.e., instead of series one takes the polynomials

$$\Pi^{(1)}(s, 2m; h) = \alpha^{(0)}(s) + h^2\alpha^{(2)}(s) + \ldots + h^{2m}\alpha^{(2m)}(s),$$

$$\Pi^{(2)}(s, 2m; h) = \beta^{(0)}(s) + h^2\beta^{(2)}(s) + ... + h^{2m}\beta^{(2m)}(s), \tag{24}$$

$$\Pi^{(3)}(s, 2m; h) = \gamma^{(0)}(s) + h^2\gamma^{(2)}(s) + ... + h^{2m}\gamma^{(2m)}(s),$$

then what integral order of accuracy in h will the difference scheme constructed with the aid of the stencils $\Pi^{(j)}(s, 2m; h)$ $(j = 1, 2, 3)$ have? We shall call such a scheme a truncated difference scheme of rank $2m$.

For the time being we shall consider the first boundary-value problem

$$^{2m}L_h^{(p,q,f)} y_i \equiv \frac{1}{h^2}\Delta\left(\frac{\nabla y_i}{^{2m}A_i^h}\right) - {}^{2m}D_i^h y_i + {}^{2m}\Phi_i^h = 0, \qquad y_0 = \mu_1, \quad y_N = \mu_2. \tag{25}$$

The coefficients of the truncated scheme (25) are determined by (18)–(20); one must replace α, β, γ by the stencil functions $\Pi^{(j)}$ $(j = 1, 2, 3)$.

Theorem. The truncated scheme of rank $2m$ has, for the first boundary-value problem (25), the $(2m + 2)$-nd integral order of accuracy, if the coefficients $p(x)$, $q(x)$, $f(x)$ of the differential equation (1) belong to the class of piecewise-continuous functions Q_0, and moreover

$$0 < K_1 \le p(x) \le K_2, \qquad 0 \le q(x) \le K_2, \qquad |f(x)| \le K_2.$$

This theorem is easily generalized to the case of boundary conditions of the 2nd and 3rd kind, if one introduces the notion of a truncated difference boundary condition of rank $2m$. From the theorem, in particular, it follows that the truncated scheme of rank zero has the 2nd integral order of accuracy in Q_0 (cf. [(3)]). Truncated schemes are a useful apparatus for constructing discrete schemes of higher order of accuracy.

§ 7. If the difference grid $S_N = \{x_0 = 0,\ x_1, \dots,\ x_{i-1},\ x_i, \dots, \dots,\ x_N = 1\}$ is nonuniform and $h = \max h_i$, $h_i = x_i - x_{i-1}$, then in this case as well an exact three-point scheme and truncated schemes of arbitrary rank can be constructed, for which the theorem of § 6 holds.

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Note: Figure translations are in progress. See original paper for figures.