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Abstract

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ON SUBSEMIGROUPS OF FREE SEMIGROUPS

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The main content of the present note is necessary and sufficient conditions for a subsemigroup of a free semigroup itself to be free.

§ 1. **Definitions and simplest properties.** Let two sets of symbols A and B be given, at least one of which is nonempty. We shall denote the symbols of the set A by $a_1, a_2, \dots, a_\alpha, \dots$, and the symbols of the set B by $b_1, b_2, \dots, b_\beta, \dots$. To the symbols of the set A we put into one-to-one correspondence certain new symbols $a_1^{-1}, a_2^{-1}, \dots, a_\alpha^{-1}, \dots$, the set of which we denote by A^{-1}. Every finite ordered sequence of symbols from the union of the sets A, A^{-1}, and B will, as usual, be called a word, if in it there do not occur adjacent to each other any symbol a_α and the corresponding symbol a_α^{-1}. All well-known notions relating to words, such as: length of a word, reductions, reduced form, the empty word, etc., we understand in the usual sense, and therefore we do not give the corresponding definitions.

If on the set of all words constructed from the symbols A, A^{-1}, and B one defines a multiplication operation consisting in appending one word to another (with, possibly, several reductions then having to be made), then this operation will be associative, as is proved in the same way as for free groups. The semigroup of words Γ obtained in this manner will be called the free semigroup generated by the symbols A, A^{-1}, and B, $\Gamma = \{A, A^{-1}, B\}$. The symbols from A, A^{-1}, and B are called the free generators of Γ.

It is clear that such a construction generalizes, on the one hand, the notion of a free group ([1]), when the set B is empty, and, on the other hand, when A is empty, the notion of a free semigroup as a semigroup of words without invertible elements (see, for example, ([2])); in accordance with the terminology adopted in ([3]), it is natural to call the latter semigroup a pure free semigroup, and a semigroup in which both sets A and B are nonempty a mixed free semigroup.

If α is the cardinality of the set A, and β is the cardinality of the set B, then the ordered pair (α, β) will be called the rank of the free semigroup $\Gamma = \{A, A^{-1}, B\}$. It is easy to see that every free semigroup is completely determined by its rank, i.e. two free semigroups are isomorphic if and only if their ranks coincide. A free semigroup of rank $(\alpha, 0)$ is a free group of rank α, and a free semigroup of rank $(0, \beta)$ is a pure free semigroup of rank β.

Let us formulate, in the form of lemmas, some of the simplest properties of a free semigroup, some of which will be needed later.

Lemma 1. *The free semigroup $\Gamma = \{A, A^{-1}, B\}$ can be embedded in a group, in particular in a free group of rank equal to the cardinality of the set $A \cup B$.*

It follows from this lemma that every subsemigroup of a free semi-

of the group has all the common properties of a semigroup lying in a group (see ([3])); in particular, the following holds:

Lemma 2. Let H be an arbitrary subsemigroup of the free semigroup $\Gamma = \{A, A^{-1}, B\}$. Then the subsemigroup $F = H \cap \{A, A^{-1}\}$ is the largest subgroup in H (the kernel of H), and the pure part $H \setminus F$ is the largest ideal of the semigroup H.

Lemma 3. In a free semigroup, for any of its elements x, a, b, each of the equalities $xa = xb$ and $ax = bx$ implies $a = b$.

Let Γ be an arbitrary semigroup. A system of generators M of this semigroup is called a **basis*** if for every $x \in M$ one has $x \notin \{M \setminus x\}$. It is clear that not every semigroup has a basis. On the other hand, the existence of a basis in a semigroup does not imply its uniqueness. The set $\Gamma \setminus \Gamma^2$, i.e. the set of indecomposable elements of the semigroup Γ, obviously belongs to any of its bases. In some cases the basis of a semigroup is exhausted by the indecomposable elements; it will then be unique. We shall now note one of these cases.

Every semigroup has the chain of its ideals

$$\Gamma \supseteq \Gamma^2 \supseteq \Gamma^3 \supseteq \cdots \supseteq \Gamma^n \supseteq \cdots.$$

Put $\Gamma^\omega = \bigcap \Gamma^n$ and call the semigroup Γ an ω-semigroup if Γ^ω is empty.

Lemma 4. Every ω-semigroup has a unique basis, consisting of indecomposable elements.

Lemma 5. Every subsemigroup of the pure free semigroup is an ω-semigroup, and consequently has a unique basis.

Lemma 6. Every free semigroup has a basis. In particular, the basis will be the set of its free generators.

Lemma 7. The pure part of a free semigroup is an ω-semigroup.

In connection with Lemmas 5 and 7 it is interesting to note that the pure part of not every subsemigroup of a free semigroup will be an ω-semigroup. A confirmation of this is provided, for example, by the following example of a subsemigroup

$$H = \{ba_1^{-1},\, ba_2^{-1},\, \dots,\, ba_k^{-1},\, \dots,\, a_2a_1^{-1},\, a_3a_2^{-1},\, \dots,\, a_{k+1}a_k^{-1},\, \dots\}$$

in the free semigroup

$$\Gamma = \{a_1, a_2, \dots, a_1^{-1}, a_2^{-1}, \dots, b\}.$$

It can be shown that H is a pure semigroup and has no basis.

Lemma 8. If the pure part of a subsemigroup H of a free semigroup is an ω-semigroup, then H has a basis.

Let H be a subsemigroup of a free semigroup, and F its kernel. A basis M of the subsemigroup H will be called **proper** if $M \cap F$ is a free basis of F.

Lemma 9. In a free semigroup every proper basis is free.

§ 2. **Theorem on free subsemigroups.** A subgroup of a free group is free; the analogous assertion about subgroups of free semigroups no longer holds, as is shown by the following trivial example: in the infinite cyclic semigroup $\{a\}$, i.e. the free semigroup of rank $(0, 1)$, the subsemigroup $H = \{a^2, a^3\}$ is not free, since the elements a^2 and a^3, which constitute the unique basis of H, are connected by the nonidentical defining relation

$$(a^2)^3 = (a^3)^2.$$

In connection with this, it is of interest to find conditions under which a subsemigroup of a free semigroup is itself free.

Let H be a subsemigroup of the free semigroup Γ having a basis. We shall call an element $s \in \Gamma$ an H**-element** under the following conditions: 1) in H there is an element h such that $sh \in H$ and $hs \in H$; 2) if, moreover, $s^{-1} \in H$, then there exist such irreducible in H representations $h = b_1 \dots b_k$ and $s^{-1} = b'_1 \dots b'_l$ of the elements h and s^{-1} through elements of some proper basis of the semigroup H, that at least one of the inequalities $b_1 \neq b'_1$, $b_k \neq b'_l$ holds.

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* For commutative semigroups, see the notion of a basis in ([4]).

A subsemigroup H of a free semigroup Γ will be called **closed** if its pure part is an ω-subsemigroup and outside H in Γ there are no H-elements.

Theorem. *A subsemigroup H of a free semigroup Γ is itself free if and only if H is closed.*

Remark 1. If Γ is a pure free semigroup, then, naturally, in the definition of an H-element one need retain only the first requirement.

Remark 2. The brevity of the formulation of our theorem was ensured by introducing the definition of a closed subsemigroup, which broke up the approach to the formulation. However, in the case when Γ is a pure free semigroup, the need for additional definitions disappears, and the formulation of the theorem (as well as its proof) is simplified by virtue of Remark 1 and Lemma 7:

A subsemigroup H of a pure free semigroup Γ is free if and only if, for all elements $s \in \Gamma$ and $h \in H$, from $sh \in H$ and $hs \in H$ it follows that $s \in H$.

Lack of space does not permit us to give here the proof of the theorem in the general case. We shall give the proof of the theorem for pure free semigroups, which is simple and transparent. The proof for the general case is based on the same idea and is carried out by an analogous method, though it becomes more complicated because of the need to take possible cancellations into account.

Necessity. Let H be a free subsemigroup of a pure free semigroup Γ. By Lemma 5 it has a unique basis consisting of elements indecomposable in H, and this basis is free. We shall denote its elements by the letters b, c, d with indices. Suppose $sh \in H$ and $hs \in H$ for some elements $s \in \Gamma$ and $h \in H$. We shall show that $s \in H$. Let

$$h = b_1 b_2 \dots b_k, \quad sh = c_1 c_2 \dots c_l, \quad hs = d_1 d_2 \dots d_m$$

be the unique expressions of the elements h, sh, and hs in terms of the free generators of H. Then

$$hsh = b_1 b_2 \dots b_{kc} 1 c_2 \dots c_l = d_1 d_2 \dots d_{mb} 1 b_2 \dots b_k,$$

whence

$$b_1 = d_1, \dots, c_l = b_k.$$

It is clear that $k < m$, since if $k > m$ then

$$b_1 b_2 \dots b_k = d_1 d_2 \dots d_{mc} 1 \dots c_{k-m},$$

i.e.

$$h = hsc_1 \dots c_{k-m},$$

and if $k = m$,

$$b_1 \dots b_k = d_1 \dots d_m,$$

i.e. $h = hs$; however, each of the equalities obtained is impossible in a pure free semigroup, since, for example, the length (in Γ) of the word h is less than the length of hs. Consequently,

$$d_1 \dots d_m = b_1 \dots b_{kc} 1 \dots c_{m-k},$$

i.e.

$$hs = hc_1 \dots c_{m-k},$$

whence, by Lemma 3,

$$s = c_1 \dots c_{m-k} \in H.$$

Sufficiency. Let H be such a subsemigroup of a pure free semigroup Γ that from $sh \in H$ and $hs \in H$, for any elements $s \in \Gamma$ and $h \in H$, it follows that $s \in H$. We shall show that H is a free subsemigroup. H has a unique basis $b_1, b_2, \dots$; consequently, it is necessary to show that it is free. Suppose the contrary, i.e. that there is a nonidentical defining relation

$$b_1 \dots b_k = b_{k+1} \dots b_{k+l},$$

where, by Lemma 3, without loss of generality one may assume

$$b_1 \neq b_{k+1}.$$

Let

$$h = a_1 \dots a_m$$

be the unique expression of the word

$$h = b_1 \dots b_k$$

in terms of the free generators of the subsemigroup Γ. Since Γ is a pure subsemigroup, i.e. no cancellations occur under multiplication, it is clear that

$$b_1 = a_1 \dots a_p, \quad b_{k+1} = a_1 \dots a_q,$$

where $p \neq q$, for otherwise we would have $b_1 = b_{k+1}$. Suppose, for example, $p > q$, whence it follows that

$$b_1 = b_{k+1} a_{q+1} \dots a_p.$$

Put

$$s = a_{q+1} \dots a_p, \quad h_1 = b_2 \dots b_{k b_{k+1}}.$$

Since b_1 is indecomposable, $s \notin H$. It is easy to see that

$$s h_1 = b_{k+2} \dots b_{k+l} b_{k+1} \in H$$

and

$$h_1 s = b_2 \dots b_{kb} 1 \in H,$$

whence, by the condition, it should follow that $s \in H$. The contradiction obtained proves the sufficiency of the theorem.

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Note: Figure translations are in progress. See original paper for figures.