DECOMPOSITION OF A REACTIVE FOUR-TERMINAL NETWORK INTO A CHAIN OF SIMPLEST FOUR-TERMINAL NETWORKS

Z(p)=

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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DECOMPOSITION OF A REACTIVE FOUR-TERMINAL NETWORK INTO A CHAIN OF SIMPLEST FOUR-TERMINAL NETWORKS

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1. In questions of synthesis of electrical four-terminal networks, an important role is played by the "reactance" theorem, according to which the matrix

$$Z(p) = \begin{pmatrix} z_{11}(p) & z_{12}(p) \\ z_{21}(p) & z_{22}(p) \end{pmatrix}$$

can be physically realized as the matrix of total resistances of a passive four-terminal network with purely reactive elements, if it satisfies the following conditions:

1) The elements of the matrix $Z(p)$ are odd rational functions of p $(p = -i\omega)$ with real coefficients.
2) The real part $\operatorname{Re} Z(p)$ of the matrix $Z(p)$ is Hermitian nonnegative in the right half-plane $(\operatorname{Re} Z(p) \geq 0,\ \text{if}\ \operatorname{Re} p \geq 0)$.

To construct the corresponding four-terminal network, the matrix $Z(p)$ is decomposed into elementary fractions. Each fraction is realized physically, and then the elementary four-terminal networks thus obtained are connected in a certain manner.

Fig. 1

However, in those cases where we are interested in the process of energy transfer inside the four-terminal network, it is desirable that it be realized in the form of a chain composed of simplest, further indecomposable, four-terminal networks, as indicated in Fig. 1.

Suppose that the voltages and currents at the terminals of the four-terminal network vary according to the law $E = Ue^{-i\omega t}$, $i = Ie^{-i\omega t}$. Denote by

$$\mathbf{x}_1 = \begin{pmatrix} U_1 \\ I_1 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} U_2 \\ I_2 \end{pmatrix}$$

the vectors whose components are respectively equal to the complex amplitudes of the voltage and current at the input and at the output. Between $\mathbf{x}_1$ and $\mathbf{x}_2$ there exists a linear relation:

$$\mathbf{x}_2 = S(\omega)\mathbf{x}_1,$$

where

$$S(\omega) = \begin{pmatrix} s_{11}(\omega) & s_{12}(\omega) \\ s_{21}(\omega) & s_{22}(\omega) \end{pmatrix}$$

is a matrix whose elements depend on the frequency ω.

We shall call the matrix $S(\omega)$ the **transmission matrix of the four-terminal network**. To a chain connection of four-terminal networks-

corresponds to a decomposition of the matrix $S(\omega)$ into factors $S(\omega) = S_n(\omega) \cdots S_2(\omega) S_1(\omega)$, where $S_1(\omega), S_2(\omega), \ldots, S_n(\omega)$ are the transmission matrices of the individual links of the chain.

2. In what follows we shall consider only passive four-terminal networks with purely reactive elements.

The transmission matrices of such four-terminal networks must satisfy the following requirements, which follow from the reactance theorem, if one takes into account the formulas relating the matrices $S(\omega)$ and $Z(p)$:

1) The elements of the matrix $S(\omega)$ have the form

$$\begin{aligned} s_{11}(\omega) &= f_{11}(\omega^2), & s_{12}(\omega) &= i\omega f_{12}(\omega^2), \\ s_{21}(\omega) &= i\omega f_{21}(\omega^2), & s_{22}(\omega) &= f_{22}(\omega^2), \end{aligned} \tag{1}$$

where $f_{11}, f_{12}, f_{21}, f_{22}$ are rational functions with real coefficients.

2) If J denotes the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then the matrix $J - S^*(\omega) J S(\omega) \geq 0$ for $\operatorname{Im}\omega \geq 0$, and for real values of ω the equality $J = S^*(\omega) J S(\omega)$ holds.

Requirement 2) means that the power supplied to the terminals of a passive four-terminal network cannot be negative. Using geometrical terminology, one may say that the matrix $S(\omega)$ is nonexpanding in the metric defined by the indefinite form

$$(x, x) = x^* J x = 2\,\mathrm{Re}(U I^*).$$

An arbitrary matrix $S(\omega)$ satisfying conditions 1) and 2) will be called **reactive**.

Thus, the problem of realizing a reactive four-terminal network in the form of a chain leads to the decomposition of the S-matrix into the simplest reactive factors. In the more general case, the problem of factorizing a J-nonexpanding matrix arose in connection with the theory of non-self-adjoint operators * in the construction of triangular models of linear operators ($^{3-5}$) and was first solved in the works of V. P. Potapov. The results of V. P. Potapov were then improved in the works of Yu. P. Ginzburg, who gave simple formulas for separating off elementary factors. In the present work we rely substantially on these results.

However, difficulties arise along this path connected with the fact that the elementary factors separated off in the cited works do not satisfy the conditions (1) of reactivity and, except for certain special cases, about which we shall speak below, cannot be physically realized.

Let ω_0 be a pole of order ν of the matrix $S(\omega)$. By virtue of Theorem 9, proved in paper (^{3}), the matrix $S(\omega)$ can be represented in the form $S(\omega) = \widetilde{S}(\omega)\xi_0(\omega)$, where $S_0(\omega)$ is a factor of the form

$$\xi_0(\omega) = I - \frac{i}{\omega - \omega_0}\Pi^*\Pi J;$$

$\Pi = (a,\ b)$ is a constant one-row matrix, with $\Pi J \Pi^* = -2\operatorname{Im}\omega_0$. The matrix $\widetilde{S}(\omega)$ is J-nonexpanding in the upper half-plane with a pole of order $\nu - 1$ at the point $\omega = \omega_0$. In the case $\omega_0 = \infty$ the factor separated off has the form

$$\xi_0(\omega) = I + i\omega\Pi^*\Pi J,$$

with $\Pi J \Pi^* = 0$.

* The connection of the theory of four-terminal networks with the theory of non-self-adjoint operators is explained by the fact that a four-terminal network is an open physical system connected to external sources by terminals. It can be shown that the distribution of currents and voltages inside the four-terminal network is determined by a non-self-adjoint operator whose characteristic matrix-function coincides with the matrix $S(\omega)$.

For what follows it is convenient to distinguish five cases: 1) the pole $\omega_0 = 0$; 2) the pole $\omega_0 = \infty$; 3) the pole ω_0 is real and nonzero; 4) the pole ω_0 is purely imaginary; 5) the pole ω_0 satisfies the condition $\operatorname{Im}\omega_0 \cdot \operatorname{Re}\omega_0 \neq 0$.

It can be proved that in cases 1) and 2) the elementary factor $\zeta_0(\omega)$ is a reactive matrix, which is realized in the form of one of the two-ports 1—4 in Fig. 2.

From relations (1) it follows that in cases 3) and 4) the reactive matrix, in addition to the pole ω_0, has the pole $-\omega_0$. Thus the factor $\zeta_0(\omega)$, having only one pole ω_0, cannot be a reactive matrix. In these cases one can always split

Fig. 2

Figure 2: Fig. 2

off from the matrix $S(\omega)$ one more factor $\bar{\zeta}_0(\omega)$, corresponding to the pole $-\omega_0$, so that the product $\bar{\zeta}_0(\omega)\zeta_0(\omega)$ is a reactive matrix. This matrix corresponds to the two-ports 5–7 in Fig. 2; moreover, for real ω_0 in circuit 7 the transformation coefficient $k > 0$, while for purely imaginary ω_0 the coefficient $k < 0$.

Fig. 2

In case 5), in addition to the pole ω_0, the matrix $S(\omega)$ must, as follows from (1), have the poles $-\omega_0$, $+\bar{\omega}_0$, $-\bar{\omega}_0$. Splitting off from the matrix $S(\omega)$ a group of four elementary factors corresponding respectively to the poles $\pm\omega_0$, $\pm\bar{\omega}_0$, one can obtain a reactive matrix whose physical realization is shown in Fig. 2, 8. Running successively through all the poles of the matrix $S(\omega)$, we obtain a representation of $S(\omega)$ in the form

$$S(\omega) = U \prod_{j=1}^{n} S_j(\omega),$$

where $S_j(\omega)$ $(j = 1, 2, \dots, n)$ are the simplest reactive matrices, each of which corresponds to one of the two-ports shown in Fig. 2, and U is a constant matrix of the form

$$U \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & k \end{pmatrix}.$$

The matrix U corresponds to an ideal transformer.

Following W. Cauer, we shall say that two two-ports are equivalent if their transfer matrices coincide for all values of the frequency ω.

Thus the following basic theorem has been proved:

Theorem. *An arbitrary reactive two-port can be replaced by an equivalent chain of the simplest indecomposable two-ports listed in Fig. 2; moreover, in the last link an ideal transformer with transfer matrix U may be needed.*

Let us note that all the two-ports listed in Fig. 2 are necessary, and none of them can in general be omitted.

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