MATHEMATICS

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Abstract

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SOME QUESTIONS ON THE APPROXIMATION OF ALMOST-PERIODIC FUNCTIONS WITH BOUNDED SPECTRUM

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1. We denote by S the class of almost-periodic functions $f(x)$ (here and below uniformly almost-periodic functions are meant) and introduce the corresponding Fourier series:

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\Lambda_k x}$$

$$(\Lambda_0 = 0; \quad \Lambda_{k+1} < \Lambda_k \text{ for } k > 0; \quad \lim_{k\to\infty} \Lambda_k = 0; \quad \Lambda_{-k} = -\Lambda_k; \quad A_0 = 0; \quad (1)$$
$$|A_k| + |A_{-k}| > 0 \quad \text{for } k \neq 0).$$

We denote by $L = L(f)$ the sequence $\{\Lambda_k\}$ $(k = 1, 2, \ldots)$.

We shall say that an almost-periodic function $f(x)$ belongs to the class S_n $(n = 1, 2, \ldots)$, if there exist functions $f_0(x), f_1(x), \ldots, f_n(x)$ possessing the following properties: $f_0(x) = f(x)$, $f'_{m+1}(x) = f_m(x)$ $(m = 0, 1, \ldots, n-1)$, $f_m(x) \in S$ $(m = 0, 1, \ldots, n)$. The inclusions

$$S \subset S_1 \subset S_2 \subset \cdots$$

are obvious.

Put

$$R_\varepsilon(f) = \operatorname{Sup}_x \left| f(x) - \sum_{|\Lambda_k| > \varepsilon} A_k e^{i\Lambda_k x} \right|,$$

$$e_\varepsilon(f) = \operatorname{Inf}_{c_k} \left\{ \operatorname{Sup}_x \left| f(x) - \sum_{|\Lambda_k| > \varepsilon} c_k e^{i\Lambda_k x} \right| \right\}, \qquad E_\varepsilon(f) = \operatorname{Inf}_{F(x)\in Q_\varepsilon} \left\{ \operatorname{Sup}_x |f(x) - F(x)| \right\},$$

where Q_ε is the class of almost-periodic functions whose Fourier exponents $\{\lambda_k\}$ satisfy the condition $|\lambda_k| > \varepsilon$.

Let

$$\Omega_f(N) = \begin{cases} \mathrm{Sup}_{|T|\geq N} \left\{ \mathrm{Sup}_x \left| \frac{1}{T} \int_0^T f(x+t)\,dt \right| \right\}, & N > 0, \\ \mathrm{Sup}\,|f(x)|, & N = 0; \end{cases}$$

$\Omega_f(N)$ is a continuous, bounded, nonincreasing function. For every function $f(x) \in S$,

$$\lim_{N\to\infty} \Omega_f(N) = 0.$$

2. **Theorem 1.** If $f(x) \in S$, then

$$e_\varepsilon(f) \leq C_0 \Omega_f\left(\frac{1}{\varepsilon}\right), \tag{2}$$

where C_0 is an absolute constant.

Proof. Put

$$\varphi_\varepsilon(t) = \begin{cases} 1, & 0 \leq |t| \leq \varepsilon, \\ 1 - 6\left(1 - \frac{|t|}{\varepsilon}\right)^2 - 6\left(1 - \frac{|t|}{\varepsilon}\right)^3, & \varepsilon < |t| < \frac{3\varepsilon}{2}, \\ 2\left(2 - \frac{|t|}{\varepsilon}\right)^3, & \frac{3\varepsilon}{2} < |t| \leq 2\varepsilon, \\ 0, & |t| > 2\varepsilon; \end{cases}$$

then

$$\Psi_\varepsilon(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_\varepsilon(t) e^{-iut}\,dt = \frac{12}{\pi}\,\frac{8\sin^3\frac{\varepsilon u}{4}\sin\frac{5\varepsilon u}{4} + \sin\varepsilon u\left(2\sin\frac{\varepsilon u}{2} - \varepsilon u\right)}{\varepsilon^3 u^4},$$

$$\rho_\varepsilon(f,x) = \int_{-\infty}^{\infty} f(x+u)\Psi_\varepsilon(u)\,du \sim \sum_{|\Lambda_k|\leq\varepsilon} A_k e^{i\Lambda_k x} + \sum_{\varepsilon<|\Lambda_k|\leq 2\varepsilon} \varphi_\varepsilon(\Lambda_k) A_k e^{i\Lambda_k x}.$$

It is easy to see that

$$e_\varepsilon(f) \leq \mathrm{Sup}_x\,|\rho_\varepsilon(f,x)|. \tag{3}$$

Integrating by parts, we obtain

$$\rho_\varepsilon(f,x) = -\int_{-\infty}^{\infty} F(u,x)\Psi_\varepsilon'(u)\,du, \qquad \text{where } F(u,x) = \int_0^u f(x+t)\,dt.$$

For any real c the inequality holds

$$\operatorname{Sup}_x |F(cu,x)| \le (2+|c|)|u|\,\Omega_f(|u|),$$

therefore

$$\rho_\varepsilon(f,x) \le C_0 \Omega_f\left(\frac{1}{\varepsilon}\right), \tag{4}$$

where

$$C_0 = \frac{12}{\pi}\int_{-\infty}^{\infty}(2+|v|)\left|\left[\frac{8\sin^3\frac{v}{4}\sin\frac{5v}{4}+\sin v\,(2\sin\frac{v}{2}-v)}{v^4}\right]'\right| dv.$$

From (3) and (4) follows the estimate (2) to be proved.

Corollary. If there exist constants A and α $(0<\alpha\le 1)$ such that

$$\left|\int_0^u f(x+t)\,dt\right| < A|u|^{1-\alpha}, \tag{5}$$

then

$$e_\varepsilon(f) \le \text{const}\cdot\varepsilon^\alpha. \tag{6}$$

Proof. In view of (5), $\Omega_f\left(\frac{1}{\varepsilon}\right) \le A\varepsilon^\alpha$. Note that the estimate (6) follows from the results of the paper ([1]) (see also ([2])).

Theorem 2. If $f(x)\in S_n$, then

$$e_\varepsilon(f) \le C_n \varepsilon^n \Omega_{f_n}\left(\frac{1}{\varepsilon}\right), \tag{7}$$

where C_n is a constant depending only on n.

The proof of the theorem is carried out by the induction method and is based on the following lemma.

Lemma. If $f(x)\in S_1$, then

$$e_\varepsilon(f) \le C\varepsilon\,\operatorname{Sup}|f_1(x)|,$$

where

$$C = \frac{12}{\pi} \int_{-\infty}^{\infty} \left| \left[\frac{8\sin^3 \frac{v}{4} \sin \frac{5v}{4} + \sin v \left(2\sin \frac{v}{2} - v\right)}{v^4} \right]' \right| dv.$$

In the estimate (7), $C_n = C_0 C^n$.

In view of the obvious inequality $e_\varepsilon(f) \geqslant E_\varepsilon(f)$, in estimates (2), (6), (7) one may replace $e_\varepsilon(f)$ by $E_\varepsilon(f)$.

3. The following theorem gives an estimate of the deviation of the partial sums of the Fourier series from an almost-periodic function of class S.

Theorem 3. Let $0 < \eta < \varepsilon,\ f(x) \in S$. Then

$$R_\varepsilon(f) \leqslant 2E_\varepsilon(f) \left\{ 1 + \frac{2}{\pi} + N_L(\eta) - N_L(\varepsilon) + \frac{1}{\pi} \ln \frac{\varepsilon + \eta}{\varepsilon - \eta} \right\}, \tag{8}$$

where

$$N_L(\varepsilon) = \sum_{\Lambda_k \geqslant \varepsilon} 1.$$

Proof. Let

$$\varphi_{\eta,\varepsilon}(t) = \begin{cases} 1, & |t| < \eta, \\ \dfrac{1}{\varepsilon - \eta}(\varepsilon - |t|), & \eta \leqslant |t| \leqslant \varepsilon, \\ 0, & |t| > \varepsilon; \end{cases}$$

then

$$\Psi_{\eta,\varepsilon}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{\eta,\varepsilon}(t) e^{-iut}\, dt = \frac{2\sin \frac{\varepsilon-\eta}{2} u\, \sin \frac{\varepsilon+\eta}{2} u}{\pi(\varepsilon - \eta) u^2},$$

$$f_{\eta,\varepsilon}(x) = \int_{-\infty}^{\infty} f(x+u) \Psi_{\eta,\varepsilon}(u)\, du \sim \sum_{|\Lambda_k| < \eta} A_k e^{i\Lambda_k x} + \sum_{\eta < |\Lambda_k| \leqslant \varepsilon} \varphi_{\eta,\varepsilon}(\Lambda_k) A_k e^{i\Lambda_k x}.$$

It is easy to see that

$$f(x) - f_{\eta,\varepsilon}(x) = \sum_{\eta < |\Lambda_k| \leqslant \varepsilon} A_k \left[1 - \varphi_{\eta,\varepsilon}(\Lambda_k)\right] e^{i\Lambda_k x} + \sum_{|\Lambda_k| > \varepsilon} A_k e^{i\Lambda_k x},$$

therefore

$$R_\varepsilon(f) \leqslant \text{Sup}_x |f_{\eta,\varepsilon}(x)| + 2 \max_{\eta<|\Lambda_k|\leqslant\varepsilon} |A_k|[N(\eta) - N(\varepsilon) + 1]. \tag{9}$$

Let $\varepsilon_1 = 0$; there exists a function $F^*(x) \in Q_\varepsilon$ such that

$$|f(x) - F^*(x)| \leqslant E_\varepsilon(f) + \varepsilon_1.$$

Then

$$\left|\int_{-\infty}^{\infty} f(x+u)\Psi_{\eta,\varepsilon}(u)\,du - \int_{-\infty}^{\infty} F^*(x+y)\Psi_{\eta,\varepsilon}(u)\,du\right| \leqslant$$

$$\leqslant [E_\varepsilon(f) + \varepsilon_1] \int_{-\infty}^{\infty} |\Psi_{\eta,\varepsilon}(u)|\,du \leqslant 2[E_\varepsilon(f) + \varepsilon_1] \left(\frac{2}{\pi} + \frac{1}{\pi} \ln \frac{\varepsilon+\eta}{\varepsilon-\eta}\right),$$

and, by the arbitrariness of ε_1,

$$|f_{\eta,\varepsilon}(x)| \leqslant 2E_\varepsilon(f) \left(\frac{2}{\pi} + \frac{1}{\pi} \ln \frac{\varepsilon+\eta}{\varepsilon-\eta}\right). \tag{10}$$

For $|\Lambda_k| \leqslant \varepsilon$,

$$A_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T [f(x) - F^*(x)] e^{-i\Lambda_k x}\,dx,$$

hence $|A_k| \leqslant E_\varepsilon(f) + \varepsilon_1$, and, since ε_1 is arbitrary,

$$\eta \leqslant \max_{|\Lambda_k|<\varepsilon} |A_k| \leqslant E_\varepsilon(f). \tag{11}$$

From (9), (10), and (11) follows (8).

In applications of Theorem 3, the choice of the parameter η is determined to a known extent (see (4)) by the character of the sequence $L(f)$.

4. Let us consider some applications of the estimates obtained above.

Theorem 4. *Let $\theta(x)$ be nonincreasing for $x \geqslant 0$, $\lim_{x\to\infty} \theta(x) = 0$, and*

$$\left|\frac{1}{u} \int_0^u f(x+t)\,dt\right| < \theta(|u|). \tag{12}$$

Then the series (1) converges uniformly, if

$$\lim_{n\to\infty} \theta\left(\frac{1}{\Lambda_n}\right) \ln \frac{\Lambda_n + \Lambda_{n+1}}{\Lambda_n - \Lambda_{n+1}} = 0.$$

Proof. In consequence of (12), $\Omega_f(N) \leqslant \theta(N)$; applying (2) and (8) with $\varepsilon = \Lambda_n$, $\eta = \Lambda_{n+1}$, we obtain

$$R_{\Lambda_n}(f) \leqslant 2C_0\theta\left(\frac{1}{\Lambda_n}\right)\left(2 + \frac{2}{\pi} + \frac{1}{\pi}\ln\frac{\Lambda_n + \Lambda_{n+1}}{\Lambda_n - \Lambda_{n+1}}\right),$$

which proves the theorem.

Theorem 5. *Let there exist a constant A such that*

$$\left|\int_0^u f(x+t)\,dt\right| < A. \tag{13}$$

Then the series (1) converges uniformly, if

$$\Lambda_n \ln \frac{\Lambda_n + \Lambda_{n+1}}{\Lambda_n - \Lambda_{n+1}} = O(1). \tag{14}$$

Proof. By virtue of (13), $f(x) \in S_1$; from (7) and (8) it follows that

$$R_{\Lambda_n}(f) \leqslant 2C_1\Lambda_n\Omega_{f_1}\left(\frac{1}{\Lambda_n}\right)\left(2 + \frac{2}{\pi} + \frac{1}{\pi}\ln\frac{\Lambda_n + \Lambda_{n+1}}{\Lambda_n - \Lambda_{n+1}}\right),$$

and, in consequence of (14),

$$\lim_{n\to\infty} R_{\Lambda_n}(f) = 0.$$

From Theorem 4, with $\theta(x) = \dfrac{1}{x^\alpha}$ $(0 < \alpha \leqslant 1)$, there follows the convergence criterion of B. M. Levitan ([1,2]). Theorem 5 is a refinement of this criterion for $\alpha = 1$.

Theorem 6. *If there exist a natural number m and $\theta > 1$ such that*

$$\frac{\Lambda_n}{\Lambda_{n+m}} \geqslant \theta \quad (n = 1, 2, ...), \tag{15}$$

then the series (1) converges uniformly.

Proof. From (15) it follows (see ([3])) that $N_L(\varepsilon) - N_L(2\varepsilon) = O(1)$. Putting in (8) $\varepsilon = \Lambda_n$, $\eta = \frac{1}{2}\Lambda_n$, we obtain $R_{\Lambda_n}(f) = O[E_{\Lambda_n}(f)]$.

Corollary. *Under the condition of Theorem 6, the series (1) converges absolutely* ([4,5])*; moreover, the order equalities hold*

$$E_\varepsilon(f) \sim R_\varepsilon(f) \sim \alpha_\varepsilon(f),$$

where

$$\alpha_\varepsilon(f) = \sum_{|\Lambda_k| \leqslant \varepsilon} |A_k|.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.