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Abstract

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THEOREMS ON AN IMPLICIT ABSTRACT ANALYTIC FUNCTION

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Let Y, Φ, and Z be complex Banach spaces. We shall call a function $\Omega(y, \varphi)$, defined in a domain D of the space $Y \times \Phi$ and taking values in Z, analytic in D with respect to the variables y and φ if it is single-valued, continuous as a function of the point $(y, \varphi)^*$, and G-differentiable** with respect to y and with respect to φ in D.

Developing the ideas of the monograph ([1]), one can show that, in a domain of the space $Y \times \Phi$ containing the point $(0, 0)$, a function $\Omega(y, \varphi)$ analytic with respect to the variables y and φ is representable in the form of an F-power series

$$\Omega(y, \varphi) = \sum_{i,j=0}^{\infty} P_{ij}^{(\Omega)} = \sum_{i,j=0}^{\infty} P_{ij}(y, \varphi),$$

where***

$$P_{ij}^{(\Omega)} = P_{ij}(y, \varphi) = \frac{\partial^i}{\partial \zeta_1^i} \frac{\partial^j}{\partial \zeta_2^j} \Omega(\zeta_1 y, \zeta_2 \varphi) \Big|_{\substack{\zeta_1 = 0 \\ \zeta_2 = 0}}$$

in some nonempty open bicircular**** C-star around $(0, 0)$ in the space $Y \times \Phi$; moreover, there exists a complete bicircular domain K with center at $(0, 0)$ (and with "absolute quadrant" K_{abs}) such that, for $(\|y\|, \|\varphi\|) \in K_{\text{abs}}$, this F-power series converges absolutely.

Let $y(\varphi)$ be a function, analytic in some domain containing 0 of the space Φ and taking values in Y; let

$$y(\varphi) = \sum_{i=0}^{\infty} y_i(\varphi)$$

be its F-power series, and let $\{a_i\}$ be a set of nonnegative numbers such that $\|y_i\| \leq a_i$. If the series

$$x(\lambda)=\sum_{i=0}^{\infty}a_i\lambda^i$$

has a nonzero radius of convergence, we shall write $y(\varphi)\preccurlyeq x(\lambda)$. Analogously we define the symbol $\preccurlyeq$ also for analytic functions of two variables; namely, we shall write $\Omega(y,\varphi)\preccurlyeq F(x,\lambda)$ if

$$\Omega(y,\varphi)=\sum_{i,j=0}^{\infty}P_{ij}(y,\varphi);\qquad F(x,\lambda)=\sum_{i,j=0}^{\infty}a_{ij}x^i\lambda^j\quad\text{and}\quad\sup_{\|y\|=1,\ \|\varphi\|=1}\|P_{ij}(y,\varphi)\|=\|P_{ij}^{(\Omega)}\|\le a_{ij}.$$

* As a norm in the space $Y\times\Phi$ one may take, for example, $\|y,\varphi\|_{Y\times\Phi}=\sqrt{\|y\|_Y^2+\|\varphi\|_\Phi^2}$.

** Here and below we use the terminology of E. Hille ([1]).

*** It is easy to see that $P_{ij}(y,\varphi)$ are polynomials continuous in $Y\times\Phi$ and homogeneous in y and φ (of order

**** By a bicircular C-star around (y_0,φ_0) we here mean a set of points $L\subseteq Y\times\Phi$,

satisfying the conditions: 1) $(y_0,\varphi_0)\in L$; 2) from $(y,\varphi)\in L$ and $|\zeta_1|\le 1$, $|\zeta_2|\le 1$

it follows that $[y_0+\zeta_1(y-y_0),\ \varphi_0+\zeta_2(\varphi-\varphi_0)]\in L$.

Lemma 1. Suppose: 1) $y(\varphi)\preccurlyeq x(\lambda)$; 2) $x(0)=0$; 3) $\Omega(y,\varphi)\preccurlyeq F(x,\lambda)$. Then

$$\Omega[y(\varphi),\varphi]\preccurlyeq F[x(\lambda),\lambda].$$

Lemma 2. Suppose: 1) $F(x,\lambda)=\sum_{i,j=0}^{\infty}a_{ij}x^i\lambda^j$ is a series convergent in some complete bicircular domain K with center at $(0,0)$; 2) $a_{ij}\geqslant 0$, $a_{00}=a_{10}=0$.

Then there exists a domain S, containing $(0,0)$, in the space of complex variables x,λ, and a unique function $x(\lambda)$ such that, for $(0,\lambda)\in S$: 1) $x(\lambda)=F[x(\lambda),\lambda]$; 2) $[x(\lambda),\lambda]\in S$.

Moreover: 1) $x(\lambda)$ is a function analytic in a neighborhood of 0:

$$x(\lambda)=\sum_{i=0}^{\infty}a_i\lambda^i;\tag{1}$$

2) $a_i \geqslant 0,\ a_0 = 0$; 3) the radius of convergence of the series (1) is determined by the formula

$$\Lambda = \sup_{x>0,(x,0)\in K} \lambda,$$

where x and λ are related by the equation $x = F(x,\lambda)$; if $F(x,\lambda) \not\equiv F(0,\lambda)$, then the series (1) also converges for $\lambda = \Lambda$; 4) as the domain S^* one may take the bicircular cylinder $|\lambda| < \Lambda,\ |x| < x(\Lambda)$.

Theorem 1. Suppose: 1) $\Omega(y,\varphi)$ is a function analytic in the variables y and φ in some domain of the space $Y \times \Phi$, containing the point $(0,0)$, and with values in Y; 2) $P_{00}^{(\Omega)} = P_{10}^{(\Omega)} = 0$.

Then there exists a domain D, containing the point $(0,0)$, of the space $Y \times \Phi$, and a unique function $y(\varphi)$ such that from $(0,\varphi) \in D$ it follows that: 1) $y(\varphi) = \Omega[y(\varphi),\varphi]$; 2) $[y(\varphi),\varphi] \in D$.

Moreover: 1) $y(\varphi)$ is a function analytic in some domain of the space Φ containing 0; 2) if

$$F(x,\lambda) = \sum_{i,j=0}^{\infty} \left\|P_{ij}^{(\Omega)}\right\| x^i \lambda^j$$

and $x(\lambda)$ is the analytic solution of the equation $x = F(x,\lambda)$, satisfying the condition $x(0) = 0$, then $y(\varphi) \preccurlyeq x(\lambda)$.

Corollary 1. $y(0) = 0$.

Corollary 2. The open C-star about 0 in Φ, where the function $y(\varphi)$ is representable in the form of an F-power series, contains the sphere $\|\varphi\| < \Lambda$, where Λ is the radius of convergence of the series $x(\lambda)$.

Corollary 3. As the domain D one may take, for example, the double C-star about the point $(0,0)$: $\|\varphi\| < \Lambda,\ \|y\| < x(\Lambda)$.

Corollary 4. There exists a unique solution continuous at the point 0 of the equation $y = \Omega(y,\varphi)$, satisfying the condition $y(0) = 0$; it will be analytic.

The theorem is proved by applying Lemmas 1 and 2 and constructing a majorant series.

Theorem 2. Let $\widetilde{Y}$ be a finitely open set in Y containing 0; let $\widetilde{\Phi}$ be some neighborhood of 0 in the space Φ, and let $\Omega(y,\varphi)$ be a function with values in Z, defined in $\widetilde{Y} \times \widetilde{\Phi}$ and satisfying the conditions: 1) $\Omega(0,0) = 0$; 2) $\Omega(y,\varphi)$ is G-differentiable with respect to y at the point $(0,0)$,

* It is easy to see that the largest of the domains S is the totality of all points (x,λ) satisfying the conditions $|\lambda|<\Delta,\ |x|<x_2(|\lambda|)$, where $x_2(\lambda)$ is the second root of the equation $x=F(x,\lambda)$. (The equation $x=F(x,\lambda)$ has, generally speaking, two roots for $|\lambda|<\Delta$. The exceptional case is when $F(x,\lambda)$ does not depend on x; in this case one should take $x_2(\lambda)=+\infty$.)

where the linear operator $I=\Omega'_y(0,0)^*$, mapping $\widetilde{Y}$ onto Z, has a bounded inverse I^{-1}; 3) $\Omega_1(z,\varphi)=\Omega(I^{-1}z,\varphi)$ is a function analytic in the variables z and φ in some domain of the space $Z\times\Phi$ containing the point $(0,0)$.

Then there exists a domain D of the space $Y\times\Phi$, containing the point $(0,0)$, and a unique function $y(\varphi)$ (with values in $\widetilde{Y}$) such that from $(0,\varphi)\in D$ it follows that: 1) $\Omega[y(\varphi),\varphi]=0$; 2) $[y(\varphi),\varphi]\in D$.

Moreover: 1) $y(\varphi)$ is a function analytic in some domain of the space Φ containing 0; 2) if

$$F(x,\lambda)=\sum_{i,j=0}^{\infty}\left\|P_{ij}^{(\Omega_1)}\right\|x^i\lambda^j$$

and $x(\lambda)$ is the analytic solution of the equation $x=\frac{1}{2}F(x,\lambda)$, satisfying the condition $x(0)=0$, then $y(\varphi)\preccurlyeq\|I^{-1}\|x(\lambda)$.

Theorem 2 has a number of corollaries similar to the corollaries of Theorem 1.

Theorem 3. Suppose: 1) $\widetilde{Y}$ is some linear manifold of the space Y, which is the domain of definition of the functions $L(y),A(y),\omega(y)$; 2) $L(\widetilde{Y})\subset Z,\ A(\widetilde{Y})\subset\Phi,\ \omega(\widetilde{Y})\subset Z;\ L(0)=0,\ A(0)=0$; 3) there exists a unique function $y=B(z,\varphi)$, with values in $\widetilde{Y}$, satisfying the equations $L(y)=z,\ A(y)=\varphi$; it is analytic in the variables z and φ in some domain of the space $Z\times\Phi$ containing the point $(0,0)$; 4) $\omega B(z,\varphi)$ is a function analytic in z and φ in some domain of the space $Z\times\Phi$ containing the point $(0,0)$, and $P_{00}^{(\omega B)}=P_{10}^{(\omega B)}=0$.

Then there exists a domain D of the space $Y\times\Phi$, containing the point $(0,0)$, and a unique function $y(\varphi)$ (with values in $\widetilde{Y}$) such that, from $(0,\varphi)\in D$, it follows that: 1) $L[y(\varphi)]=\omega[y(\varphi)],\ A[y(\varphi)]=\varphi$; 2) $[y(\varphi),\varphi]\in D$.

Moreover: 1) $y(\varphi)$ is a function analytic in some domain of the space Φ containing 0; 2) if

$$F(x,\lambda)=\sum_{i,j=0}^{\infty}\left\|P_{ij}^{(\omega B)}\right\|x^i\lambda^j,\qquad f(x,\lambda)=\sum_{i,j=0}^{\infty}\left\|P_{ij}^{(B)}\right\|x^i\lambda^j$$

and $x(\lambda)$ is the analytic solution of the equation $x=F(x,\lambda)$, satisfying the condition $x(0)=0$, then $y(\varphi)\preccurlyeq f[x(\lambda),\lambda]$.

The theorem has corollaries similar to the corollaries of Theorems 1 and 2.

From the qualitative part of our theorems follow the classical theorems: Weierstrass' s theorem on an implicit analytic function; Poincaré' s theorem on the expansion, in a power series in the parameter, of the solution of a differential equation; Lyapunov' s theorem on the holomorphy of the solution of a system of differential equations with respect to the initial conditions, as well as a number of other results (in particular, the results of Hildebrandt and Graves (2)). The quantitative part of the theorems makes it possible to estimate the domains of convergence and the remainders of series for all the nonlinear problems listed above, and also for many others.

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2. T. H. Hildebrandt, L. M. Graves, *Trans. Am. Math. Soc.*, **29**, 127 (1927).

* No boundedness requirement is imposed on the operator I.

Note: Figure translations are in progress. See original paper for figures.