Reports of the Academy of Sciences of the USSR

D. D. STANCU

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1960. Volume 134, No. 1

MATHEMATICS

D. D. STANCU

ON SOME POLYNOMIALS OF TWO VARIABLES OF BERNSTEIN TYPE AND SOME OF THEIR APPLICATIONS

(Presented by Academician A. N. Kolmogorov on 15 IV 1960)

Recently, considerable interest has been attracted by the study of the polynomials

$$B_n(f;x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right),$$

introduced in 1912 by S. N. Bernstein (^{1}), both because of their remarkable properties and because of their numerous applications to a variety of problems in mathematics.

Recently results have been obtained in the problem of approximation of functions of two variables by Bernstein-type polynomials

$$B_{m,n}(f;x,y) = \sum_{i=0}^{m} \sum_{k=0}^{n} \binom{m}{i} \binom{n}{k} x^i (1-x)^{m-i} y^k (1-y)^{n-k} f\left(\frac{i}{m}, \frac{k}{n}\right).$$

(See $^{2-7}$.)

In note (^{8}) we presented some results concerning the approximation of a function $f(x,y)$, defined and continuous on the triangle $\Delta : x+y \leq 1,\ x \geq 0,\ y \geq 0$, using Bernstein-type polynomials of degree n with respect to both variables:

$$B_n(f;x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_n^{i,j}(x,y) f\left(\frac{i}{n}, \frac{j}{n}\right), \tag{1}$$

where

$$p_n^{i,j}(x,y) = \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j}.$$

The polynomials (1) are obtained with the aid of Steffensen' s interpolation polynomial of two variables. These polynomials may be arrived at by a simpler route, starting from certain probabilistic considerations. For this purpose, let us perform n trials under identical conditions, each of which can lead to one of the incompatible events E_1, E_2, E_3. If the corresponding probabilities are denoted by x, y, z, then $x + y + z = 1$. The probability that in these n trials the event E_1 occurred i times, the event E_2 j times, and the event E_3 $k = n - i - j$ times is equal to

$$\frac{n!}{i!\, j!\, k!} x^i y^j z^k = p_n^{i,j}(x,y).$$

If, for example, we agree to pay to a certain player, carrying out n trials, the sum $f\left(\frac{i}{n}, \frac{j}{n}\right)$, if among all the trials he realizes the first event i times, the second j times, and the third k times, then the corresponding mathematical expectation is equal to $B_n(f; x, y)$.

In the paper (^{8}) we indicated the order of approximation on the triangle Δ of the function $f(x,y)$ by means of the polynomial (1), namely

$$|f(x,y) - B_n(f; x,y)| \leqslant 2\omega\left(\frac{1}{\sqrt{n}}\right),$$

where $\omega(\delta)$ is the modulus of continuity of the function $f(x,y)$

$$\omega(\delta) = \max |f(x_2,y_2) - f(x_1,y_1)|,$$

and (x_1,y_1), (x_2,y_2) are points of Δ satisfying the inequality $|x_2 - x_1| + |y_2 - y_1| \leqslant \delta > 0$.

It was also shown that if $f(x,y)$ has on Δ a continuous partial derivative $\partial^{r+s} f(x,y)/\partial x^r \partial y^s$, then the inequality

$$\left| \frac{\partial^{r+s} f(x,y)}{\partial x^r \partial y^s} - \frac{\partial^{r+s} B_n(f; x,y)}{\partial x^r \partial y^s} \right| \leqslant$$

$$\leqslant (1 + \sqrt{1 + 2r + 2s})\, \omega_{r,s}\left(\frac{1}{\sqrt{n - r - s}}\right) + \frac{(r+s)(r+s-1)}{2n} M_{r,s},$$

holds, where $0 \leqslant r + s < n$, $\omega_{r,s}(\delta)$ is the modulus of continuity of the indicated partial derivative,

$$M_{r,s} = \max_{\Delta} \left| \frac{\partial^{r+s} f(x,y)}{\partial x^r \partial y^s} \right|.$$

It follows from this that on Δ, uniformly,

$$\lim_{n\to\infty} \frac{\partial^{r+s} B_n(f;x,y)}{\partial x^r \partial y^s} = \frac{\partial^{r+s} f(x,y)}{\partial x^r \partial y^s}.$$

We point out one generalization of certain important results of E. V. Voronovskaya (^{9}) and S. N. Bernstein (^{10}). Let p and q be nonnegative integers. For the particular case of functions $f(t,r) = n^{p+q}(t-x)^p(r-y)^q$, we obtain

$$B(f(t,r);x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} (i-nx)^p (j-ny)^q p_n^{i,j}(x,y) \equiv S_{p,q}^{(n)}(x,y). \tag{2}$$

These polynomials play an important role in the investigation of the error of approximation by the polynomials $B_n(f;x,y)$ of a function $f(x,y)$ admitting continuous derivatives on Δ up to some order.

For computing the polynomial (2), certain recurrence formulas have been established. Let

$$g(t,r;x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} e^{(i/n-x)t+(j/n-y)r} p_n^{i,j}(x,y).$$

It is easy to show that

$$g(t,r;x,y) = \sum_{\nu=0}^{\infty}\sum_{k=0}^{\nu} \frac{1}{n^\nu} \frac{S_{\nu-k,k}^{(n)}(x,y)}{(\nu-k)!\,k!} t^{\nu-k} r^k =$$

$$= e^{-xt-yr}\left[1 + x\left(e^{t/n}-1\right) + y\left(e^{r/n}-1\right)\right]^n. \tag{3}$$

Using the identity

$$\frac{\partial g}{\partial t} = x\left(e^{t/n}-1\right)\left[(1-x)g - \frac{\partial g}{\partial t}\right] - y\left(e^{r/n}-1\right)\left[xg + \frac{\partial g}{\partial t}\right]$$

and formula (3), we directly establish the recurrence relation

$$S_{p+1,q}^{(n)}(x,y) = x\sum_{i=1}^{p}\binom{p}{i}\left[n(1-x)S_{p-i,q}^{(n)}(x,y) - S_{p+1-i,q}^{(n)}(x,y)\right] -$$

$$- y \sum_{j=1}^{q} \binom{q}{j} [nxS^{(n)}_{p,q-i}(x,y) + S^{(n)}_{p+1,q-j}(x,y)].$$

In the same way one can arrive at a second recurrence relation, of a form analogous to this one. The initial values are

$$S^{(n)}_{0,0}(x,y) = 1, \qquad S^{(n)}_{1,0}(x,y) = S^{(n)}_{0,1}(x,y) = 0.$$

Applying the method proposed by I. P. Natanson (11) in establishing a certain recurrence relation for polynomials of one variable corresponding to the polynomials (2), one can establish the following recurrence relations:

$$S^{(n)}_{p+1,q}(x,y) = x(1-x)\left[\frac{\partial}{\partial x}S^{(n)}_{p,q}(x,y) + npS^{(n)}_{p-1,q}(x,y)\right] -$$

$$- xy\left[\frac{\partial}{\partial y}S^{(n)}_{p,q}(x,y) + nqS^{(n)}_{p,q-1}(x,y)\right],$$

$$S^{(n)}_{p,q+1}(x,y) = y(1-y)\left[\frac{\partial}{\partial y}S^{(n)}_{p,q}(x,y) + nqS^{(n)}_{p,q-1}(x,y)\right] -$$

$$- xy\left[\frac{\partial}{\partial x}S^{(n)}_{p,q}(x,y) + npS^{(n)}_{p-1,q}(x,y)\right].$$

If $1 < p+q \leq 3$, then we obtain the polynomials

$$S^{(n)}_{2,0}(x,y) = nx(1-x), \qquad S^{(n)}_{1,1}(x,y) = -nxy, \qquad S^{(n)}_{0,2}(x,y) = ny(1-y),$$

$$S^{(n)}_{3,0}(x,y) = nx(1-x)(1-2x), \qquad S^{(n)}_{2,1}(x,y) = -nxy(1-2x),$$

$$S^{(n)}_{1,2}(x,y) = -nxy(1-2y), \qquad S^{(n)}_{0,3}(x,y) = ny(1-y)(1-2y).$$

The polynomials (2) can be represented in the form

$$S^{(n)}_{p,q}(x,y) = \sum_{i=0}^{\left[\frac{p+q}{2}\right]} A^{(i)}_{pq}(x,y)n^i,$$

where $A^{(i)}_{p,q}(x,y)$ are polynomials in x and y, independent of n. Then on Δ we have

$$\left|S_{p,q}^{(n)}(x,y)\right| \leq C(p,q) n^{\left[\frac{p+q}{2}\right]},$$

where $C(p,q)$ does not depend on n.

On the basis of the results obtained, it is proved that:

If the function $f(x,y)$ admits on Δ bounded partial derivatives of order N (≥ 2), then the asymptotic equality holds

$$B_n(f;x,y) = f(x,y) + \sum_{\nu=}^{N} \frac{1}{n^{\nu}} \frac{S_{\nu-k,k}^{(n)}(x,y)}{(\nu-k)!\,k!} f_{x^{\nu-k}y^{k}}^{(\nu)}(x,y) + \frac{\varepsilon_n}{n^s},$$

where

$$s = \left[\frac{N+1}{2}\right],$$

and ε_n tends to zero as $n \to \infty$.

In the particular case $N = 2$ the formula reduces to

$$B_n(f;x,y) = f(x,y) + \frac{x(1-x)}{2n} f_{x^2}''(x,y) - \frac{xy}{n} f_{xy}''(x,y) +$$

$$+ \frac{y(1-y)}{2n} f_{y^2}''(x,y) + \frac{\varepsilon_n}{n},$$

which is a generalization to two variables of the formula of E. V. Voronovskaya (^{9}).

In the case of a function $f(x,y)$ admitting continuous derivatives on Δ up to a certain order, one can, using the results obtained, construct polynomials analogous to (1), but converging to the function $f(x,y)$ more rapidly.

Babeș-Bolyai University
Cluj, Romania

Received
30 I 1960

CITED LITERATURE

[1] S. N. Bernstein, *Communications of the Kharkov Mathematical Society*, **13**, 1 (1912).
[2] E. H. Kingsley, *Proc. Am. Math. Soc.*, **2**, No. 1, 64 (1951).
[3] I. A. Kiprianov, *Scientific Notes of Kazan State University*, **113**, 193 (1953).
[4] P. L. Butzer, *Canad. J. Math.*, **5**, 107 (1953).

[5] A. F. Ipatov, *Scientific Notes of Petrozavodsk University*, **4**, No. 4, 31 (1957).
[6] A. F. Ipatov, *Scientific Notes of Petrozavodsk University*, **4**, No. 4, 49 (1957).
[7] T. Popoviciu, *Desfre cea mai bună aproximație a funcțiilor continue prin polinoame*, Cluj, 1937.
[8] D. D. Stancu, *Com. Acad. R. P. R.*, **9**, No. 8, 773 (1959).
[9] E. V. Voronovskaya, DAN, A, 79 (1932).
[10] S. N. Bernstein, DAN, A, 86 (1932).
[11] I. P. Natanson, *Constructive Theory of Functions*, Moscow-Leningrad, 1949.

Note: Figure translations are in progress. See original paper for figures.