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Abstract

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Some Inequalities for Differentiable Functions of Several Variables

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In the present note we consider some inequalities for functions belonging to the space $W_p^{(l)}$ of S. L. Sobolev ([1]) in a domain D of n-dimensional space, where l is any positive real number, $p \geqslant 1$. In the case of nonintegral l, such spaces were introduced by L. N. Slobodetskii ([2,3]).

Let us note that all the results follow from integral representations of functions analogous to the well-known identity of S. L. Sobolev ([1]).

I. Let D be a domain of n-dimensional space satisfying the condition: for any two points X and Y from D, for which $|X-Y| \leqslant \mathcal{H}$, where $\mathcal{H}$ is a fixed number independent of X and Y, there exist n-dimensional spherical sectors of the same aperture and of radius $\leqslant |X-Y|$, entirely contained in D, such that the measure of the common part of these sectors is $\geqslant \lambda |X-Y|^n$, where $\lambda > 0$ is a constant number independent of X and Y. The class of domains of this kind will be denoted by $C_H(\lambda)$.

We introduce the following notation. By D_m we shall denote an arbitrary section of the domain D by the hyperplane $x_{m+1} = \text{const}, \dots, x_n = \text{const}$. Let D_m be some section of the domain D by the hyperplane $x_{m+1} = a_{m+1}, \dots, x_n = a_n$. Let s be an integer, with $0 \leqslant s \leqslant m$, and let D_s be the s-dimensional section $x_{s+1} = a_{s+1}, \dots, x_m = a_m,\ x_{m+1} = a_{m+1}, \dots, x_n = a_n$; then by $[D_s]^d_{m-s}$ we shall denote the set of points $X(x_1, \dots, x_s, x_{s+1}, \dots, x_m, a_{m+1}, \dots, a_n)$ of the section D_m, for whose coordinates the inequalities $|x_i - a_i| \leqslant d$ $(i = s+1, \dots, m)$ hold. In particular, for example, $[D_m]^d_0$ coincides with D_m, while $[D_0]^d_m$ is the part of D_m contained inside the m-dimensional cube with edge $2d$.

II. In what follows we shall assume that $f(x_1, \dots, x_n)$ is a continuous function defined in a domain $D \in C_H(\lambda)$, having continuous derivatives up to order $\bar{l} = [l]$, where $[l]$ is the integral part of l, and satisfying the conditions:

$$1) \qquad \left[\underbrace{\int \overset{n}{\cdots} \int}_{(D)} |f(X)|^p \, dX \right]^{1/p} \leqslant A \qquad (p \geqslant 1); \tag{1}$$

2) there exists a constant number $M > 0$ such that for any integer m, $0 \leqslant m \leqslant n$, and any $d > 0$ the inequality holds:

$$\sup_{D_m} \sum_{i_1,\dots,i_{\bar{l}}=1}^{n} \left[\underbrace{\int \overset{n}{\cdots} \int}_{[D_m]^d_{n-m}} \left(\underbrace{\int \overset{n}{\cdots} \int}_{(D)} \frac{\left| \frac{\partial^{\bar{l}} f(X)}{\partial x_{i_1} \cdots \partial x_{i_{\bar{l}}}} - \frac{\partial^{\bar{l}} f(Y)}{\partial x_{i_1} \cdots \partial x_{i_{\bar{l}}}} \right|^p}{|X-Y|^{n+(l-\bar{l})p}} dY \right) dX \right]^{1/p} \leqslant M d^{\alpha_m}, \tag{2}$$

if l is noninteger, and

$$\sup_{D_m} \sum_{i_1,\dots,i_l=1}^{n} \left[\int_{[D_m]_{n-m}} \cdots \int \left| \frac{\partial^l f(X)}{\partial x_{i_1} \cdots \partial x_{i_l}} \right|^p dX \right]^{1/p} \leq M d^{\alpha_m}, \tag{2'}$$

if l is an integer, where α_m $(m = 0, 1, \dots, n)$ are fixed numbers satisfying the inequalities

$$\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_n = 0, \qquad \alpha_m \leq \frac{n-m}{p}. \tag{3}$$

From the fact that $\alpha_n = 0$ it follows that, for $d \geq 1$, in the right-hand side of inequalities (2), (2′) the factor d^{α_m} may be omitted.

Theorem 1. Let $f(X)$ satisfy conditions (1)–(3) in $D \in C_H^\lambda$, and let k be an integer, with $0 \leq k < l$.

Then:

1) If $\varepsilon_0 = l + \alpha_0 - n/p$, $\varepsilon_0 - k > 0$, $0 < \beta \leq \varepsilon_0 - k$, $\beta \leq 1$, then the inequalities hold:

$$\left| \frac{\partial^k f(X)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \leq C_1 A h^{-k-n/p} + C_2 M h^{\varepsilon_0 - k}, \tag{4}$$

$$\frac{\left| \frac{\partial^k f(X)}{\partial x_{i_1} \cdots \partial x_{i_k}} - \frac{\partial^k f(Y)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|}{|X-Y|^\beta} \leq \begin{cases} C_3 \left(A h^{-k-n/p-\beta} + M h^{\varepsilon_0 - k - \beta} \right), & \text{if } \varepsilon_0 - k > 1 \text{ or } \varepsilon_0 - k < 1; \\ C_4 \left(A h^{-k-n/p-\beta} + \dfrac{1}{1-\beta} M h^{\varepsilon_0 - k - \beta} \right), & \text{if } \varepsilon_0 - k = 1,\ \beta < 1; \\ C_5 \left[A H^{-k-n/p-\beta} + M \left(1 + \left| \ln \frac{\mathcal{H}}{|X-Y|} \right| \right) \right], & \text{if } \varepsilon_0 - k = 1,\ \beta = 1, \end{cases} \tag{5}$$

where h is an arbitrary positive number $\leq \mathcal{H}$; C_i are constants independent of A, M, h.

2) If $0 < \beta < 1$, $q \geq p$, m is an integer, $1 \leq m \leq n$, $\varepsilon_m = l + \alpha_0(1 - p/q) + \alpha_m p/q + m/q - n/p$, $\varepsilon_m - k - \beta > 0$, s is an integer, with $0 \leq s \leq m$, H is an

arbitrary positive fixed number, $\delta = l+\alpha_0(1-p/q)+\alpha_s p/q+s/q-n/p-k$, then the inequalities hold:

$$I_1 = \left[\int_{[D_s^H]_{m-s}} \cdots \int \left|\frac{\partial^k f(X)}{\partial x_{i_1} \cdots \partial x_{i_k}}\right|^q dv_m\right]^{1/q} \leq \tag{6}$$

$$\leq \begin{cases} \text{a) } C_6\left(AH^{(m-s)/q-\nu}h^{s/q+\nu-n/p-k} + MH^{(m-s)/q-\mu}h^{\delta+\mu}\right), \\ \qquad \text{if } \delta > 0;\ \nu,\mu \text{ are arbitrary numbers satisfying } 0 \leq \nu,\mu \leq \dfrac{m-s}{q}. \\ \text{b) } C_7\left[AH^{(m-s)/q-\nu}h^{s/q+\nu-n/p-k} + MH^{(m-s)/q-(\alpha_s-\alpha_m)p/q-\mu}h^{\mu}\left(1+\left|\ln\frac{h}{H}\right| \times H^{(\alpha_s-\alpha_m)p/q}\right)\right], \\ \qquad \text{if } \delta = 0,\ \text{where } \nu,\mu \text{ are arbitrary, satisfying} \\ \qquad 0 \leq \nu \leq (m-s)/q, \quad 0 \leq \mu \leq (m-s)/q-(\alpha_s-\alpha_m)p/q. \\ \text{c) } C_8\left(AH^{(m-s)/q-\nu}h^{s/q+\nu-n/p-k} + MH^{(m-s)/q-(\alpha_s-\alpha_m)p/q+\delta-\mu}h^{\mu}\right), \\ \qquad \text{if } \delta < 0, \end{cases}$$

where $0 \leqslant \nu \leqslant (m-s)/q$, $0 \leqslant \mu \leqslant (m-s)/q-(\alpha_s-\alpha_m)p/q+\delta$ (α_s, as in δ, $\max u$ in all inequalities may be replaced by α_m);

$$I_2 = \left[\int_{(D_m)} \cdots \int_{(D_m)} \left(\int_{(D_m)} \cdots \int_{(D_m)} \frac{\left|\dfrac{\partial^k f(X_m)}{\partial x_{i_1} \cdots \partial x_{i_k}} - \dfrac{\partial^k f(Y_m)}{\partial x_{i_1} \cdots \partial x_{i_k}}\right|^q}{|X-Y|^{m+\beta q}} dY_m\right) dX_m\right]^{1/q} \leqslant$$

$$\leqslant C_9\left(Ah^{m/q-n/p-k-\beta} + Mh^{\varepsilon_m-k-\beta}\right). \tag{7}$$

In (6) and (7) h is an arbitrary positive number $\leqslant \mathcal{H}$; C_i are constants independent of A, M, h, H, ν, μ.

Remark 1. If $\alpha_i = 0$ ($i = 1, \dots, n$), $q > p > 1$, $1 \leqslant m \leqslant n$, $0 \leqslant \beta < 1$, $l+m/q-n/p-k-\beta = 0$, then:

1) for $\beta = 0$

$$I_1 \leqslant C_{10}\left(A\mathcal{H}^{m/q-n/p-k} + M\right);$$

2) for $0 < \beta < 1$

$$I_2 \leqslant C_{11}\left(A\mathcal{H}^{m/q-n/p-k-\beta} + M\right).$$

Remark 2. If $\alpha_i = 0$ ($i = 1, \dots, n$), $\delta = s/q+l-k-n/p = 0$, then under various additional assumptions concerning s, q, and the domain D, estimate (6) can be improved.

1) If $s \geqslant 1,\ q > p > 1$, then

$$I_1 \leqslant C_{12}\left(AH^{(m-s)/q-\nu}h^{s/q+\nu-n/p-k} + MH^{(m-s)/q-\mu}h^{\mu}\right), \quad 0 \leqslant \nu, \mu \leqslant (m-s)/q.$$

2) If $s \geqslant 1,\ q = p,\ D \in C_H^{n-s}$ (each point of D is attainable by means of an $(n-s)$-dimensional sector contained in the section of the domain D by the hyperplanes $x_1 = \text{const}, \dots, x_s = \text{const}$), then

$$I_1 \leqslant C_{13}\left[AH^{(m-s)/p-\nu}h^{s/p+\nu-n/p-k} + MH^{(m-s)/p-\mu}h^{\mu}\left(1 + \left|\ln\frac{h}{H}\right|^{1/p'}\right)\right],$$

$$0 \leqslant \nu, \mu \leqslant \frac{m-s}{p}.$$

An analogous inequality for the case $l = 1,\ q = p = 2,\ s = n-2,\ m = n-1$ was obtained by S. M. Nikol' skii ([4]).

Remark 3. Put

$$\left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{W_q^{(\beta)}(D_m)} = \left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{L_q(D_m)} +$$

$$+\left[\int_{(D_m)}\cdots\int_{(D_m)}\left(\int_{(D_m)}\cdots\int_{(D_m)}\frac{\left|\dfrac{\partial^k f(X)}{\partial x_{i_1}\cdots\partial x_{i_k}} - \dfrac{\partial^k f(Y)}{\partial x_{i_1}\cdots\partial x_{i_k}}\right|^q}{|X-Y|^{m+\beta q}}\,dY\right)dX\right]^{1/q},$$

$$\left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{C^{(\beta)}(D)} = \left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{C(D)} + \sup_{X,Y\in D}\frac{\left|\dfrac{\partial^k f(X)}{\partial x_{i_1}\cdots\partial x_{i_k}} - \dfrac{\partial^k f(Y)}{\partial x_{i_1}\cdots\partial x_{i_k}}\right|}{|X-Y|^{\beta}},$$

if $0 < \beta < 1$, and

$$\left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{W_q^{(0)}(D_m)} = \left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{L_q(D_m)},$$

$$\left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{C^{(0)}(D)} = \left\|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\right\|_{C(D)},$$

if $\beta = 0$.

From inequalities (4)–(7) it follows that

$$\left\| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right\|_{C^{(\beta)}(D)} \leq C_{14} \left(\|f\|_{L_p(D)}^{\frac{\varepsilon_0 - k - \beta}{1+\alpha_0}} M^{\frac{n/p+k+\beta}{1+\alpha_0}} + \|f\|_{L_p(D)} \right), \tag{8}$$

if $\varepsilon_0 - k - \beta > 0$, and

$$\left\| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right\|_{W_q^{(\beta)}(D_m)} \leq C_{15} \left(\|f\|_{L_p(D)}^{\frac{\varepsilon_m - k - \beta}{1+\alpha_0(1-p/q)+\alpha_m p/q}} M^{\frac{n/p+k+\beta-m/q}{1+\alpha_0(1-p/q)+\alpha_m p/q}} + \|f\|_{L_p(D)} \right), \tag{9}$$

if $\varepsilon_m - k - \beta > 0$.

We note that, under certain additional assumptions concerning the domain D, all the results extend also to functions having generalized derivatives in the sense of S. L. Sobolev.

Finally, let us observe that if there is a set of functions bounded in the sense of (1)–(2), then from the fact that this set will be compact in $L_p(D')$, if D' is a bounded domain such that $\overline{D'} \subset D$, and from (8) and (9) it follows directly that the derivatives of order k form a set compact in $C^{(\beta)}(D')$, if $\varepsilon_0 - k - \beta > 0$, or, respectively, in $W_q^{(\beta)}(D'_m)$, if $\varepsilon_m - k - \beta > 0$.

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Note: Figure translations are in progress. See original paper for figures.