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**Abstract**

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**MATHEMATICS**

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# ON THE SPECTRUM OF AN OPERATOR AND ON THE DIRICHLET PROBLEM FOR THE EQUATION

$$\Box^2 u + 4\frac{\partial^2}{\partial t^2}\Box u + 2\frac{\partial^4 u}{\partial t^4} = f(x, y, z, t)$$

*(Presented by Academician S. L. Sobolev on 24 II 1960)*

Let $\Omega$ be a domain in the finite part of the four-dimensional space $XYZT$, bounded by the surface $\Gamma(x, y, z, t) \equiv x^2 + y^2 + z^2 + t^2 - 1 = 0$, the unit sphere with center at the origin.

1°. In the domain $\Omega$ consider the Hilbert space $H_B^*(\Omega)$, which is obtained by completing the linear manifold $D_B$ of infinitely differentiable functions of finite support in $\Omega$ (vanishing in some boundary strip of the domain $\Omega$) with scalar product

$$\begin{aligned}(u, v)_B = \int \cdots \int_\Omega \Big\{ & \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 u}{\partial t^2}\frac{\partial^2 v}{\partial t^2} \\ & + 2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y} + 2\frac{\partial^2 u}{\partial x\partial z}\frac{\partial^2 v}{\partial x\partial z} + 2\frac{\partial^2 u}{\partial x\partial t}\frac{\partial^2 v}{\partial x\partial t} + 2\frac{\partial^2 u}{\partial y\partial z}\frac{\partial^2 v}{\partial y\partial z} \\ & + 2\frac{\partial^2 u}{\partial y\partial t}\frac{\partial^2 v}{\partial y\partial t} + 2\frac{\partial^2 u}{\partial z\partial t}\frac{\partial^2 v}{\partial z\partial t}\Big\} d\Omega + \iiint_\Gamma u\, d\sigma \cdot \iiint_\Gamma v\, d\sigma \\ & + \iiint_\Gamma \frac{\partial u}{\partial n}\, d\sigma \cdot \iiint_\Gamma \frac{\partial v}{\partial n}\, d\sigma. \end{aligned} \tag{1}$$

For $u, v \in D_B$, (1) takes the form

$$(u, v)_B = \int \cdots \int_\Omega \Delta^2 u \cdot v\, d\Omega. \tag{1*}$$

In the space $H_B^*(\Omega)$ consider the operator $B^2$, defined by the formula $B^2 = \Delta^{-2}\dfrac{\partial^4}{\partial t^4}$, where $\Delta^{-2}$ is the operator inverse to the four-dimensional biharmonic Laplace operator under the boundary conditions

$$u\Big|_\Gamma = 0, \qquad \frac{\partial}{\partial n}\Big|_\Gamma = 0;$$

$n$ is the outward normal to the boundary surface $\Gamma$.

**Theorem 1.** *The operator $B^2$ is a symmetric bounded and positive-definite operator on the dense manifold $D_B$ of the space $H_B^*(\Omega)$.*

We shall denote the self-adjoint extension of the operator $B^2$ in $H_B^*(\Omega)$ by the same letter $B^2$.

**Theorem 2.** *The spectrum of the operator $B^2$ in $H_B^*(\Omega)$ is discrete.*

**Proof.** The scheme of the proof of this theorem is essentially analogous to the method of P. Denchev ([1]). We shall briefly outline this scheme. Let $R_n$ be the space of all polynomials of degree not exceeding $n$; let $N$ be its dimension. Introduce a scalar product in $R_n$ by the formula: $(p, q) = \int \cdots \int_\Omega pq\Gamma^2\, d\Omega$ for $p, q \in R_n$. Consider in $R_n$ the operators:

$$L_1(p) = \frac{\partial^4}{\partial t^4}(\Gamma^2 p), \qquad L_2(p) = \Delta^2(\Gamma^2 p) \tag{2}$$

for all $p \in R_n$. The operators $L_1$ and $L_2$ are symmetric and map $R_n$ into itself; moreover, the operator $L_2$ is positive. Then, by the well-known theorem from linear algebra [^1] on reducing two quadratic forms to a canonical basis, we conclude that there exist $N$ numbers $\lambda_1^2, \lambda_2^2, \dots, \lambda_N^2$ and $N$ linearly independent polynomials $\gamma_1(x, y, z, t), \gamma_2(x, y, z, t), \dots, \gamma_N(x, y, z, t)$ such that

$$L_1(\gamma_k) - \lambda_k^2 L_2(\gamma_k) = 0 \qquad (k = 1, 2, \dots, N); \tag{3}$$

$$(L_2\gamma_i, \gamma_j) = \delta_{ij} \qquad (i, j = 1, 2, \dots, N). \tag{4}$$

The polynomials

$$\chi_k(x, y, z, t) = \Gamma^2(x, y, z, t)\gamma_k(x, y, z, t) \qquad (k = 1, 2, \dots, N) \tag{5}$$

will be eigenfunctions of the operator $B^2$ in $H_B^*(\Omega)$. Assigning the values $n = 1, 2, \dots$, we obtain an infinite system of polynomial eigenfunctions for the operator $B^2$ in $H_B^*(\Omega)$. This system will be complete in $H_B^*(\Omega)$. Indeed, first note that every polynomial $\mathcal{P}(x, y, z, t)$ that vanishes on the boundary surface $\Gamma$ together with its normal derivative has the form

$$\mathcal{P}(x,y,z,t) = \Gamma^2(x,y,z,t) \cdot P(x,y,z,t). \tag{6}$$

Next, every polynomial of the form (6) is a linear combination of the polynomial eigenfunctions $\{\chi_k(x,y,z,t)\}$ of the operator $B^2$ in $H_B^*(\Omega)$. On the other hand, by polynomials of this form one can uniformly approximate, together with their derivatives, smooth finite functions in $\Omega$ which, according to the definition of the Hilbert space $H_B^*(\Omega)$, are in turn everywhere dense in $H_B^*(\Omega)$ in the sense of the convergence of this space. This proves the completeness of the polynomial eigenfunctions of the operator $B^2$ in $H_B^*(\Omega)$, and the theorem is proved.

$2^\circ$. In this paragraph we prove the possibility of applying the result obtained above to the study of the following boundary-value problem:

$$L(u) \equiv \Box^2 u + 4\frac{\partial^2}{\partial t^2}\Box u + 2\frac{\partial^4 u}{\partial t^4} = f(x,y,z,t), \tag{7}$$

$$u\Big|_\Gamma = 0, \tag{8}$$

$$\frac{\partial u}{\partial n}\Big|_\Gamma = 0, \tag{8'}$$

where

$$\Box \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$$

is the four-dimensional wave operator, and $n$ is the outward normal to the boundary surface $\Gamma$.

The following theorem holds, which shows, in a certain sense, the connection of the boundary-value problem (7), (8), (8′) with the spectrum of the operator $B^2$ in $H_B^*(\Omega)$. Here $\Omega$ is a finite domain bounded by a sufficiently smooth surface $\Gamma$.

**Theorem 3.** *If the operator $B^2$ in $H_B^*(\Omega)$ has a complete orthonormal system of eigenfunctions, then, for uniqueness of the solution of the boundary-value problem (7), (8), (8′), it is necessary and sufficient that the number $\mu^* = 1/2$ not be an eigenvalue of the operator $B^2$ in $H_B^*(\Omega)$.*

**Proof.** Necessity is obvious: indeed, if $\mu^* = 1/2$ were an eigenvalue of the operator $B^2$ in $H_B^*(\Omega)$, then the corresponding eigenfunction $u^*$ would be a nontrivial solution of the boundary-value problem; hence the boundary-value problem (7), (8), (8′) would have a nonunique solution.

Let now $\mu^* = 1/2$ not be an eigenvalue for the operator $B^2$ in $H_B^*(\Omega)$, and suppose that $u^*$ is a nontrivial solution of the boundary-value problem (7), (8), (8′). Then $u^* \in H_B^*(\Omega)$ and, consequently, can be expanded in $H_B^*(\Omega)$ in a series in the eigenfunctions $\{\chi_k\}$ $(k = 1, 2, ...)$ of the operator $B^2$, i.e.

$$u^* = \sum_{k=1}^{\infty} a_k \chi_k. \tag{9}$$

Further, from

$$a_k = (u^*, \chi_k) = \iiiint_\Omega \Delta^2 u^* \chi_k \, d\Omega = \iiiint_\Omega u^* \Delta^2 \chi_k \, d\Omega = \frac{1}{\lambda_k} \iiiint_\Omega \frac{\partial^4 u^*}{\partial t^4} \chi_k \, d\Omega,$$

$$a_k = \iiiint_\Omega \Delta^2 u^* \chi_k \, d\Omega = 2 \iiiint_\Omega \frac{\partial^4 u^*}{\partial t^4} \chi_k \, d\Omega \qquad (k = 1, 2, \dots, N) \tag{10}$$

it follows that $a_k = 0$ $(k = 1, 2, ...)$, i.e. $u^*(x, y, z, t) \equiv 0$. The theorem is proved.

Let now $f(x, y, z, t) \in W_2^{(2)}(\Omega)$. Put

$$F(x, y, z, t) = \frac{1}{6} \int_0^t (t - \tau)^3 f(x, y, z, \tau) \, d\tau. \tag{11}$$

Since $F \in W_2^{(2)}(\Omega)$, we have

$$F = F_0 + \sum_{k=1}^{\infty} F_k \chi_k, \tag{12}$$

where $F_0$ is a polynomial of order not exceeding 3.

**Theorem 4.** If the series

$$\sum_{k=1}^{\infty} \frac{F_k^2}{(1/\lambda_k - 2)^2} \tag{13}$$

converges, then the boundary-value problem has a solution in $W_2^{(2)}(\Omega)$.

**Proof.** From condition (13) of the theorem it follows that the series

$$\sum_{k=1}^{\infty} a_k \chi_k(x, y, z, t), \qquad a_k = \frac{F_k}{1/\lambda_k - 2}, \tag{14}$$

converges in $W_2^{(2)}(\Omega)$. Let

$$\sum_{k=1}^{\infty} a_k \chi_k = u^*(x, y, z, t).$$

This means that

$$S_n = \sum_{k=1}^{n} a_k \chi_k \to u^* \in W_2^{(2)}(\Omega). \tag{15}$$

We shall show that $u^*$ is a solution of the boundary-value problem. To this end, first note that the function $u^*$ satisfies the boundary conditions (8), (8′), since $u^*$ is the limit, in the sense of $W_2^{(2)}(\Omega)$, of functions from $W_2^{(2)}(\Omega)$ satisfying these boundary conditions. Further, using the embedding theorems of S. L. Sobolev ([2]) and the relation

$$\frac{\partial^4 F}{\partial t^4} = f(x, y, z, t), \tag{16}$$

one can prove that

$$\iiint\int_{\Omega} u^* L\varphi \, d\Omega = \iiint\int_{\Omega} f\varphi \, d\Omega \tag{17}$$

for all $\varphi \in \Phi_0$, where $\Phi_0$ is a linear manifold of infinitely differentiable finite functions. This means precisely that $u^*(x, y, z, t)$ is a solution of the boundary-value problem (7), (8), (8′). The theorem is proved.

3°. Let now $H$ be the Hilbert space obtained by completing $D_B$ in the sense of the scalar product

$$(u, v) = \iiint\int_{\Omega} \left\{ \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} \right\} d\Omega. \tag{18}$$

In the space $H$ consider the operator $B = \Delta^{-1}\dfrac{\partial^2}{\partial t^2}$, where $\Delta^{-1}$ is the operator inverse to the four-dimensional Laplace operator under zero boundary conditions. It can be proved[4] that the operator $B$ on the linear manifold $D_B$, dense in $H$, is a symmetric, bounded, and positive-definite operator. We shall denote the hypermaximal extension of this operator by the same letter $B$.

**Theorem 5.** *The limiting spectrum of the operator $B$ in $H$ coincides with the interval* $[0, 1]$.

**Proof.** Consider the sequence of functions

$$u_{k,n}(x,y,z,t) = \frac{T_n\left(t\cos\frac{k}{n}\frac{\pi}{2} + \sqrt{x^2+y^2+z^2}\sin\frac{k}{n}\frac{\pi}{2}\right)}{\sqrt{x^2+y^2+z^2}} + \frac{(-1)^{k+1}T_n\left(t\cos\frac{k}{n}\frac{\pi}{2} - \sqrt{x^2+y^2+z^2}\sin\frac{k}{n}\frac{\pi}{2}\right)}{\sqrt{x^2+y^2+z^2}}, \tag{19}$$

where $n \geq 2$; $k = 2, 4, \dots, n-1$; $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind of degree $n$. By direct verification[3,5] one can see that the function $u_{k,n}(x,y,z,t)$ satisfies the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \nu_{k,n}^2 \frac{\partial^2 u}{\partial t^2} = 0, \tag{20}$$

$$u|_\Gamma = 0, \tag{21}$$

where $\nu_{k,n} = \operatorname{tg}\dfrac{k}{n}\dfrac{\pi}{2}$, $n \geq 2$, and $k$ takes even values up to $n-1$. On the other hand, let us note that the solutions of the boundary-value problem (20), (21) are eigenfunctions for the operator $B$ in $H$ with eigenvalues

$$\lambda_{k,n}^2 = \frac{1}{1+\nu_{k,n}^2},$$

and since the numbers

$$\lambda_{k,n}^2 = \frac{1}{1+\operatorname{tg}^2\frac{k}{n}\frac{\pi}{2}}$$

for $n \geq 2$, $k = 2, 4, \dots, n-1$ are everywhere dense in the interval $[0,1]$, the limiting spectrum of the operator $B$ in $H$ coincides with $[0,1]$. The theorem is proved.

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*Note: Figure translations are in progress. See original paper for figures.*