

# APPROXIMATE SOLUTION OF THE PROBLEM OF THE MOTION OF A CONDUCTING PLASMA

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**Abstract**

**Full Text**

**PHYSICS**

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**APPROXIMATE SOLUTION OF THE PROBLEM OF THE MOTION OF A CONDUCTING PLASMA**

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In papers <sup>(1-4)</sup> a new method of asymptotic integration of linear partial differential equations of hyperbolic type was developed, and the application of this method to obtaining asymptotic solutions for the equation of acoustics and Maxwell's equations was shown. In papers <sup>(5-8)</sup> the indicated method was developed as applied to the solution of dynamical problems of the theory of elasticity.

The general idea of the method in the case of a linear partial differential equation of hyperbolic type (for example, the wave equation) is as follows: by a special choice of functions we try to satisfy the original equations approximately, i.e., we seek the solution in the form

$$u(x, y, z, t) = A(x, y, z) \exp\{i\omega[t - \Phi(x, y, z)]\} \quad (1)$$

under the condition that  $\omega \rightarrow \infty$ .

As a result we obtain the well-known relations:

$$\text{grad}^2 \Phi = \frac{1}{c^2}, \quad (2)$$

$$2(\text{grad} A \text{ grad} \Phi) + A\Delta\Phi = 0, \quad (3)$$

$\Phi(x, y, z)$  is the eikonal of the wave,  $A(x, y, z)$  is the amplitude of the oscillation. On the other hand, it is well known that the equation for the discontinuity jump of discontinuous solutions of wave equations coincides with equation (3). In other words, the approximate solution (1) as  $\omega \rightarrow \infty$  coincides on the wave front with the discontinuous solution that may exist for the exact solution of the original equation.

Thus, an identity is established between the discontinuity jump of a nonstationary wave front and the amplitude of the "geometrical approximation," which corresponds to the trajectories of rays orthogonal to these wave fronts.

The simplicity of the physical interpretation of the asymptotic method for the case of linear equations, unfortunately, is not preserved for quasilinear and nonlinear equations. However, formally this method also makes it possible here to solve a number of problems.

In the present work the authors apply an approximate method to the problem of integrating the equations of plasma oscillations. The problem of the motion of a gas in a medium with finite conductivity  $\sigma$  is investigated. We shall assume that the medium obeys an equation of state of the form  $P = \rho e^{\frac{S-S_0}{c_v}} \gamma$ . In this case, if we assume that  $\mathbf{v} \perp \mathbf{H}$ , where  $\mathbf{v} = (u, 0, 0)$ ;  $\mathbf{H} = (0, H, 0)$ , and neglect the displacement current, then the system of equations of magnetogasdynamics in the one-dimensional

in this case we obtain in the form <sup>(9)</sup>

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} &= 0, \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial x} \left( P + \frac{H^2}{8\pi} \right) &= 0, \\ \frac{\partial P}{\partial t} + u \frac{\partial P}{\partial x} + \gamma P \frac{\partial u}{\partial x} &= \frac{(\gamma - 1)}{4\pi} \varkappa \left( \frac{\partial H}{\partial x} \right)^2, \\ \frac{\partial H}{\partial t} + \frac{\partial u H}{\partial x} &= \varkappa \frac{\partial^2 H}{\partial x^2}. \end{aligned} \tag{4}$$

Here  $\varkappa = c^2/4\pi\sigma$  is the magnetic viscosity;  $\sigma$  is the conductivity;  $\gamma = c_p/c_v$  is the ratio of specific heats;  $P$  is the pressure;  $\rho$  is the density;  $u$  is the gas velocity;  $c$  is the speed of light;  $S$  is the entropy;  $H$  is the magnetic-field strength.

The first equation of system (4) is the continuity equation, the second is the equation of motion, the third is the energy equation, and the last is Maxwell's equation written with Ohm's law taken into account.

Thus, the problem consists in finding the unknowns  $P, \rho, H$ , and  $u$  in a sufficiently general form, i.e., in such a way that they contain arbitrary functions which can then be determined from the initial and boundary conditions.

Since the energy equation is not exact (thermal conductivity and radiation are not taken into account), while the exact equation has an excessively complicated form <sup>(10)</sup>, it is reasonable, in determining the quantities  $P, \rho, H$ , and  $u$ , to use only three equations of system (4), specifying  $u$  in a definite form up to an arbitrary function and a constant.

We proceed to the solution of the stated problem. Introducing the pseudoscalar potential  $\varphi$  by means of the relation  $H = \partial\varphi/\partial x$ , the last equation of system (4) can be written in the form

$$\frac{\partial \varphi}{\partial t} + u \frac{\partial \varphi}{\partial x} = \varkappa \frac{\partial^2 \varphi}{\partial x^2}. \quad (5)$$

We shall seek a solution of equation (5) by setting

$$\varphi = A(x, y, z, t) e^{i\omega f(x, y, z, t)}. \quad (6)$$

Substituting (6) into (5) and separating the real and imaginary parts, we find that

$$\frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} = \varkappa \left[ \frac{\partial^2 A}{\partial x^2} - \omega^2 A \left( \frac{\partial f}{\partial x} \right)^2 \right]; \quad (7)$$

$$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = \varkappa \left[ \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial f}{\partial x} \frac{\partial}{\partial x} \ln A \right]. \quad (8)$$

For  $\omega \gg 1$  one can show that

$$\frac{\partial^2 A}{\partial x^2} \ll \omega^2 A \left( \frac{\partial f}{\partial x} \right)^2; \quad (9)$$

in this case equation (7) takes the form

$$\frac{\partial}{\partial t} (\ln A) + u \frac{\partial}{\partial x} (\ln A) + \varkappa \omega^2 \left( \frac{\partial f}{\partial x} \right)^2 = 0. \quad (10)$$

In what follows we shall consider the class of solutions (6) subject to an additional condition, i.e., we shall assume that

$$\omega \sqrt{\varkappa} \frac{\partial f}{\partial x} = B = \text{const}. \quad (11)$$

(It is also possible to consider the more general case  $B = B(t)$ ; in this case the solution of equation (8) is not fundamentally complicated.)

Set  $\omega \sqrt{\varkappa} = \alpha$ , and we shall choose the order of  $\omega$  so that the order of  $\alpha$  corresponds to the order of the remaining terms of the equation. With the aid of (11) we obtain

$$f = \frac{B}{\alpha} x + T(t). \quad (12)$$

Substituting (11) into (8) and taking (12) into account, we shall have

$$u = 2\kappa \frac{\partial}{\partial x} \ln A - \frac{\alpha}{B} \dot{T}. \quad (13)$$

Eliminating  $u$  from (10) and (13), we arrive at the equation

$$\dot{T} \frac{\alpha}{B} \frac{\partial}{\partial x} \ln A - \frac{\partial}{\partial t} \ln A - B^2 = 2\kappa \left( \frac{\partial}{\partial x} \ln A \right)^2.$$

Differentiating with respect to  $x$  and denoting  $\theta = (\ln A)_x$ , we obtain

$$\frac{\partial \theta}{\partial t} + \left[ 4\kappa \theta - \frac{\alpha}{B} \dot{T} \right] \frac{\partial \theta}{\partial x} = 0. \quad (14)$$

The solution of this equation has the form

$$x = 4\kappa t \theta - \frac{\alpha}{B} T + F(\theta), \quad (15)$$

where  $F(\theta)$  is an arbitrary function.

Thus, to determine  $\theta$  and  $u$  we have two arbitrary functions  $T(t)$  and  $F(\theta)$ , and one arbitrary constant  $B$ . Consider the case when  $F(\theta) = \beta\theta$ , where  $\beta = \text{const} < 0$ . In this case we arrive at a linear function for  $u = u(x, t)$ . Indeed, from (15) we have

$$x = (4\kappa t + \beta)\theta - \frac{\alpha}{B} T. \quad (16)$$

Substituting the value of  $\theta$  from (16) into (13), we obtain:

$$u = 2\kappa \frac{x + \frac{\alpha}{B} T}{4\kappa t + \beta} - \frac{\alpha}{B} \dot{T}. \quad (17)$$

Moreover, taking (16) into account, we have:

$$A = A_0(t) \exp \left[ \frac{\frac{x^2}{2} + \frac{\alpha}{B} x T}{4\kappa T + \beta} \right]. \quad (18)$$

Consequently, our solution of equation (3) will take the form

$$\varphi = A_0(t) \exp \left[ \frac{\frac{x^2}{2} + \frac{\alpha}{B} x T}{4\kappa t + \beta} \right] \cos \omega \left( \frac{B}{\alpha} x + T \right). \quad (19)$$

Hence, for the magnetic-field strength we obtain

$$H = A_0(t) \left\{ \frac{x + \frac{\alpha}{B}xT}{4\kappa t + \beta} \cos \omega \left( \frac{B}{\alpha}x + T \right) - \frac{B\omega}{\alpha} \sin \omega \left( \frac{B}{\alpha}x + T \right) \right\} \exp \left[ \frac{\frac{x^2}{2} + \frac{\alpha}{B}xT}{4\kappa t + \beta} \right]. \quad (20)$$

Thus, one of the unknown quantities entering the system (4) has been obtained by us.

The arbitrary function  $T(t)$  can be determined by introducing some condition for  $u$ , for example, assuming that at  $x = x_0$ ,  $u = 0$  (at the wall), or, in a more general case, that the wall moves according to the law  $x = \psi(t)$ . Then  $u = \dot{x} = \dot{\psi}(t)$ . From equation (17) we obtain

$$\dot{T} = \frac{2\kappa T}{4\kappa t + \beta} - \frac{B}{\alpha} \left[ \dot{\psi} - \frac{2\kappa\psi}{4\kappa t + \beta} \right]. \quad (21)$$

or

$$T = \sqrt{4\kappa t + \beta} \left\{ \text{const} + \frac{\beta}{\alpha} \int \left[ \frac{2x\psi}{4\kappa t + \beta} - \dot{\psi} \right] \frac{dt}{\sqrt{4\kappa t + \beta}} \right\}. \quad (22)$$

The constant can be determined by assuming that at  $t = 0$ ,  $T = 0$ . If the value of  $\dot{T}$  from (21) is substituted into (17), then the latter expression takes the form

$$u = \frac{2x(x - \psi)}{4\kappa t + \beta} + \dot{\psi}. \quad (23)$$

Next, from the first equation of system (4) we find the value of the density  $\rho$ . Since, taking (23) into account,

$$\frac{2x}{4\kappa t + \beta} + \frac{\partial}{\partial t}(\ln \rho) + \left[ \frac{2x(x - \psi)}{4\kappa t + \beta} + \dot{\psi} \right] \frac{\partial}{\partial x}(\ln \rho) = 0,$$

we have

$$\rho = \sqrt{4\kappa t + \beta} \Phi(z), \quad (24)$$

where  $\Phi(z)$  is an arbitrary function, with

$$z = \frac{x}{\sqrt{4\kappa t + \beta}} + \int \left[ \frac{2x\psi}{4\kappa t + \beta} - \dot{\psi} \right] \frac{dt}{\sqrt{4\kappa t + \beta}}. \quad (25)$$

After this, from the second equation of system (4) we find the pressure  $P$ :

$$P + \frac{H^2}{8\pi} = P_0(t) - \int \rho(u_t + uu_x) dx, \quad (26)$$

where  $P_0(t)$  is an arbitrary function of time.

The arbitrary functions  $\Phi(z)$  and  $P_0(t)$  must be determined from the boundary conditions; for example, in the case of motion with a shock wave, assuming that on its front the known relations between  $\rho = \rho(u)$  and  $P = P(u)$  are satisfied.

Solving the problem for a concrete form  $\psi = \psi(t)$  presents no difficulty.

Thus, with the aid of relations (20), (23), (24), and (26), we determine all the quantities entering system (4).

In conclusion, we note that in integrating equation (7) we neglected the term  $\frac{1}{A} \frac{\partial^2 A}{\partial x^2}$  in comparison with  $\omega^2 \left(\frac{\partial f}{\partial x}\right)^2 = \frac{B^2}{x}$ . It is not hard to see that

$$\frac{x}{B^2} \frac{1}{A} \frac{\partial^2 A}{\partial x^2} = \frac{x}{B^2} \left[ \left( \frac{x + \frac{a}{B}T}{4xt + \beta} \right)^2 + \frac{1}{4xt + \beta} \right] \approx \frac{1}{\omega^2 \left(\frac{\partial f}{\partial x}\right)^2} \ll 1,$$

i.e., our assumption is justified.

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*Note: Figure translations are in progress. See original paper for figures.*

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