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Abstract

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THE CENTRAL LIMIT THEOREM FOR GEODESIC FLOWS ON MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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1°. In paper ([7]) it is shown that dynamical systems generated by geodesic flows on manifolds of constant negative curvature are arranged analogously to dynamical systems generated by regular stationary random processes ([1]). In view of this analogy, it seems natural to attempt to apply the methods found in recent years for proving the central limit theorem of probability theory for random processes of this kind to geodesic flows on manifolds of constant negative curvature.

Definition. A measurable essentially bounded real function f, given on a Lebesgue space M with measure μ ([6]), in which a measurable ergodic flow $\{S^t\}$ acts, is said to **satisfy the central limit theorem** if, for every fixed α, $-\infty < \alpha < \infty$,

$$\lim_{t\to\infty} \mu \left\{ x : \frac{\int_0^t f(S^\tau x)\, d\tau - t\bar{f}}{\sqrt{D_t(f)}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-u^2/2}\, du,$$

where

$$\bar{f} = \int_M f(x)\, d\mu \quad \text{and} \quad D_t(f) = \int_M \left[\int_0^t f(S^\tau x)\, d\tau \right]^2 d\mu.$$

The aim of the present paper is to show that, in the case where M is the space of line elements of a compact manifold of constant negative curvature, and $\{S^t\} = \{G^t\}$ is the geodesic flow in M ([4,8]), there exists a sufficiently broad class of functions satisfying the central limit theorem.

2°. **Theorem 1.** *Let an essentially bounded real function f, given on the space M of line elements of a compact manifold of constant negative curvature, satisfy the following conditions:*

1. *There exist numbers $\alpha > 0$, $\varepsilon > 0$, $\varepsilon_1 > 0$, $c_1 > 0$, $c_2 \geq 0$, such that for all*

$$\mu\left\{x : \sup_{y:\rho(x,y)<\rho^*}\left|\int_0^\alpha f(G^\tau y)\,d\tau - \int_0^\alpha f(G^\tau x)\,d\tau\right| > \frac{c_1}{|\log^{1+\varepsilon}\rho|}\right\} \leq \frac{c_2}{|\log^{4+\varepsilon_1}\rho|}$$

($\rho(x,y)$ is the metric in the space of line elements ([8])).

2. $D_t(f) \sim ct$ as $t \to \infty$, where $c > 0$.
3. *For every* $\varepsilon > 0$ *there exist such* $N(\varepsilon)$ *and* $T(\varepsilon)$ *that for all* $t > T(\varepsilon)$

$$\frac{1}{D_t(f)}\int_{\left\{x:\left|\int_0^t f(G^\tau x)\,d\tau\right|>N(\varepsilon)\sqrt{D_t(f)}\right\}}\left[\int_0^t f(G^\tau x)\,d\tau\right]^2 d\mu \leq \varepsilon.$$

Then the function f obeys the central limit theorem.

Verification of condition 1 presents no difficulty, since it depends on the local properties of the function f. We note only that it is certainly satisfied by continuous functions f having modulus of continuity of order $1/|\log^{1+\varepsilon}\rho|$, and also by functions of the form $\chi_A(x)$, where A is a set of line elements whose supports lie inside some domain A_0 of the manifold F with piecewise smooth boundary, and $\chi_A(x)$ is its characteristic function.

To verify condition 2 one must consider the Fourier transform $r(\lambda)$ of the function

$$B(t) = \int_M (f - \bar{f})(f(G^t x) - \bar{f})\,d\mu$$

(which exists, since any function $f \in \mathcal{L}^2_\mu(M)$ has, with respect to the corresponding group $\{U^t\}$ of unitary operators, a Lebesgue spectrum). If the function $r(\lambda)$ is continuous in a neighborhood of $\lambda = 0$ and $r(0) \neq 0$, then condition 2 is satisfied and the constant c occurring in it is equal to $\pi r(0)$.

Below we give a theorem which, in the case of two-dimensional manifolds F, makes it possible in some cases to establish the fulfillment of condition 3 of Theorem 1. Before stating it, let us recall some notions and facts concerning geodesic flows on a surface of constant negative curvature.

First, on a surface of constant negative curvature one can define the horocycle flow H^τ, $-\infty < \tau < \infty$, in the following way: each line element $x \in M$ is moved along the horocycle determined by it through a distance τ under the transformation H^τ. Since only orientable manifolds are considered, the direction of displacement can be specified uniquely. The horocycle flow preserves the invariant measure μ ([8]).

Second, there exist measurable partitions $\xi^0, \tilde{\xi}^0$ of the space M, almost every element of which is a set of line elements with supports on some arc of a horocycle of the surface F and with directions perpendicular to this horocycle, and

$$1)\quad \xi^t > \xi^{t_1} = G^t \xi^0, \quad \tilde{\xi}^t = G^t \tilde{\xi}^0 < \tilde{\xi}^{t_1} \quad \text{for } t > t_1;$$

$$2)\quad \prod_{-\infty}^{\infty} \xi^t = \prod_{-\infty}^{\infty} \tilde{\xi}^t = \varepsilon,$$

where ε is the partition of the space M mod 0 into individual points;

$$3)\quad \bigcap_{-\infty}^{\infty} \xi^t = \bigcap_{-\infty}^{\infty} \tilde{\xi}^t = \nu,$$

where ν is the trivial partition, the only element of which is the whole space mod 0 ([7]).

Let $f \in \mathcal{L}^2_\mu(M)$. Denote by $f_t(\tilde{f}_t)$ the projection of f onto the Hilbert space of functions in $\mathcal{L}^2_\mu(M)$ which are constant mod 0 on the elements of the partition $\xi^t(\tilde{\xi}^t)$. By property 2) of the partitions $\xi^t(\tilde{\xi}^t)$ we have

$$\lim_{t\to\infty} \|f - f^t\| = \lim_{t\to-\infty} \|f - \tilde{f}^t\| = 0.$$

Theorem 2. Let f be a measurable, essentially bounded function on the space M such that $\bar{f} = 0$, and suppose the following conditions are satisfied:

1. There exist numbers $\beta_1 > 0$, $c_1 > 0$, $c_2 > 0$ such that for $t > 0$

$$\|f - f_t\| \le c_1 e^{-c_2 t^{\beta_1}}, \qquad \|f - \tilde{f}_{-t}\| \le c_1 e^{-c_2 t^{\beta_1}}.$$

2. There exist numbers $\beta_2 > 0$, $D_1 > 0$, $D_2 > 0$ such that

$$\left| \int_M f(H^\tau x) f(x)\, d\mu \right| \le D_1 \frac{D_2}{|\tau|^{\beta_2}}.$$

Then the function f satisfies condition 3 of Theorem 1.

Let us explain the meaning of conditions 1 and 2 of Theorem 2. Condition 1 is certainly satisfied for functions f satisfying a Hölder condition of any order, and also for functions of the form $\chi_A(x)$ (see above). Further, for negative t sufficiently large in absolute value, the integral $\int_C f(x)\, d\mu(x \mid c)$ is close to zero for a set of elements C of the partition ξ^t of measure arbitrarily close to one. Condition 2 provides information on the rate at which the indicated integral tends to zero and the measure of the corresponding set of elements of the partition tends to one. The proof of the theorem consists in estimating the integral $\int_M f(G^{t_1}x) f(G^{t_2}x) f(G^{t_3}x) f(x)\, d\mu$ for $t_1 > t_2 > t_3 > 0$ as a function of $\max(t_3, t_2 - t_3, t_1 - t_2)$, from which follows the finiteness of the expression

$$\frac{1}{t^4}\int_M \left[\int_0^t f(G^\tau x)\,d\tau\right]^4 d\mu$$

for all t, sufficient for the fulfillment of condition 4 of Theorem 1.

Theorem 3. *Functions $f(x) = f(z, \theta)$, having, for each z, a derivative with respect to θ satisfying a Hölder condition of fixed order $\alpha > 0$ uniformly in z, satisfy condition 2 of Theorem 2.*

Proof. For each z, expand the function f in a Fourier series

$$f(x) = \sum_{-\infty}^{\infty} f_n(z)e^{in\theta}.$$

From the condition of the theorem it follows that there is a constant $c'' > 0$ such that $|f_n(z)| \le c''/n^{1+\alpha}$. Further,

$$\left|\int_M f(x)f(H^t x)\,d\mu\right| \le \sum_{k,l}\int_M |f_k(z)|\,|f_l(H^t z)|\,d\mu.$$

Expand each function $|f_k|$ in a series in the orthonormal eigenfunctions v_i of the Laplace operator of the surface F:

$$|f_k| = \sum_{i=1}^{\infty} c_{ki}v_i.$$

It is shown in [2] that each function v_i belongs to some irreducible representation of the group of real matrices of order 2 with determinant 1 of the principal or supplementary series. It is easy to show that there exist constants $\rho > 0$, $c''' > 0$, independent of i, such that

$$\int_M v_i(x)v_i(H^t x)\,d\mu < \frac{c'''}{|t|^\rho}.$$

Therefore

$$\left|\int_M f(x)f(H^t x)\,d\mu\right| \le \left|\sum_{k,l}\sum_i c_{ki}c_{li}\int v_i(x)v_i(H^t x)\,d\mu\right| \le \frac{c'''}{|t|^\rho}\sum_{k,l}\sum_i c_{ki}c_{li}.$$

But, by virtue of the condition of the theorem,

$$\sum_{k,l}\sum_i c_{ki}c_{li} \le \left[\sum_k \left(\sum_i |c_{ki}|^2\right)^{1/2}\right]^2 = \left[\sum_k \left(\int |f_k(z)|^2\, d\mu\right)^{1/2}\right]^2 \le \left(\sum_k \frac{c'}{k^{1+\alpha}}\right)^2 < \infty.$$

The theorem is proved.

3°. We give two examples of applications of Theorems 1, 2, and 3.

Example 1. Let $\chi_A(x) \equiv \chi_{A_0}(z)$ be the characteristic function of the set of line elements whose carriers lie inside the domain A_0

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* z is the coordinate determining the position of the carrier of the line element on the surface F; θ is the coordinate determining its direction at each point.

surface F with piecewise smooth boundary. It is obvious that it satisfies condition 1 of Theorem 1. From Theorems 2 and 3 it follows that it also satisfies condition 3 of Theorem 1. To prove that it also satisfies condition 2 of Theorem 1, let us expand it, as in the proof of Theorem 3, in terms of the eigenfunctions v_i of the Laplace operator of the surface F:

$$\chi_{A_0}(z) = \sum_{i=1}^{\infty} \alpha_i v_i.$$

We shall again use the fact that each function v_i belongs to some irreducible representation of the principal or complementary series [(2)] of the group of unimodular real matrices of order 2. Then

$$D_t(\chi_A(x)) \le \sum |\alpha_i|^2 D_t(v_i),$$

and it remains to show that $D_t(v_i) \sim c_i t$ for all i. For this, pass to the canonical realization of the function v_i and of the corresponding subspace. Then it is easy to compute that

$$|(v_i, U^t v_i)| = \left|\int_M v_i(x)\overline{v_i}(G^t x)\, d\mu\right| < \alpha_i e^{-\lambda_i |t|},$$

where U^t is the group of unitary operators corresponding to the geodesic flow, and α_i, λ_i are positive constants. In view of the last formula it is enough to establish that the equation $Ah = v_i$, where A is the infinitesimal operator of the group of unitary operators $\{U^t\}$, has no solution in the space $\mathcal{L}^2_\mu(M)$. The latter fact is proved by a simple calculation. Thus we obtain that the functions $\chi_A(x)$ obey the central limit theorem.

Example 2 is connected with the mean rotation numbers introduced by I. M. Gelfand and I. I. Pyatetskii-Shapiro [(3)]. Let $\gamma_1, \dots, \gamma_p$ be a basis of the integral homologies of the manifold M. Connect any two points x and y of the manifold M by a fixed geodesic $g(x, y)$ and expand the closed path $\gamma(t) : x \to G^t x \to g(G^t x, x)$ in the basis $\gamma_1 \dots \gamma_p$:

$$\gamma(t) = m_1(t)\gamma_1 + \dots + m_p(t)\gamma_p.$$

In [(3)] it is shown that there exist analytic functions φ_i on M such that

$$m_i(t) = \int_0^t \varphi_i(G^\tau x)\,d\tau + \int_{g(G^t x, x)} \varphi_i\,ds.$$

It is also computed there that

$$\lim_{t\to\infty} \frac{1}{t} m_i(t, x) = 0$$

for almost all x, and

$$\int_M |m_i(t, x)|^2\,d\mu \sim ct.$$

Using Theorems 1 and 2, we obtain that there exists a constant $\sigma > 0$ such that, for any α, $-\infty < \alpha < \infty$,

$$\lim_{t\to\infty} \mu\left\{x : \frac{m_i(t, x)}{\sqrt{t\sigma}} < \alpha\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-u^2/2}\,du.$$

In conclusion I express my gratitude to A. N. Kolmogorov for posing the problem and for valuable comments concerning the results given above, and also to I. I. Pyatetskii-Shapiro for a number of indications concerning Theorem 3.

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Note: Figure translations are in progress. See original paper for figures.