



Soviet-era science, translated into English

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1960

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Abstract

Full Text

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ON THE IDENTITY PROBLEM IN ASSOCIATIVE SYSTEMS OF A SPECIAL KIND

(Presented by Academician P. S. Novikov, 7 VII 1960)

1. An associative system of a special kind, given by generators

$$a_1, a_2, \dots, a_n \tag{1}$$

and defining relations

$$A_i = 1 \quad (i = 1, 2, \dots, k), \tag{2}$$

will be called an **associative system of rank l** , if all words A_i in the relations (2) have length l .

Theorem 1. *If the natural numbers k and l are such that in every group with k nontrivial defining relations*

$$B = 1 \quad (i = 1, 2, \dots, k),$$

whose left-hand sides have lengths not exceeding l , the identity problem is solvable, then in every associative system of rank l with k defining relations the identity problem is also solvable.

Let the numbers k and l satisfy the condition of Theorem 1, and let the associative system \mathfrak{B} of rank l be given by the generators (1) and the defining relations (2). In what follows, unless special reservations are made, the words considered by us will be words in the alphabet (1), and equalities of words will be equalities in the system \mathfrak{B} . By the sign \equiv we shall denote equality in the free associative system. We shall call a **left (right) wing** of a word A such a nonempty word B that $A \equiv BC$ ($A \equiv CB$) for some C . In the case when C is nonempty, the word B will be called a **proper left (right) wing** of the word A .

We shall call a word A **left-sided (right-sided)** with respect to the system of relations (2) if $A \equiv B_1 B_2 \dots B_r$, where each B_j is a left (right) wing of some word A_i from the relations (2). We shall call a word A **two-sided with respect to the system of relations (2)** if it is both left-sided and right-sided with respect to this system of relations.

2. First operation of extending the system of defining relations. If, in a word A_i from the system of relations (2), one performs a cyclic permutation

of a left or right wing B , which is a two-sided word with respect to (2), then the word thus obtained is also equal to 1 in the system \mathfrak{B} . The first operation of extending the system of defining relations (2) will consist in adding to these relations all those relations which are obtained from them by cyclic permutation of left or right wings that are two-sided words with respect to this system of relations. To the system of relations obtained as a result of one application of the first extension operation, one may again apply this operation, first introducing the notions of left-sided, right-sided, and two-sided word

with respect to this new system of relations, etc. Since all the relations newly added in this process have left-hand sides of length l , after a finite number of steps we shall obtain a system of defining relations

$$A_i = 1 \quad (i = 1, 2, \dots, k, \dots, k_1), \quad (3)$$

closed with respect to the application of the first extension operation.

A word A , two-sided with respect to the system of relations (3), whose length is $\leq l$, will be called **elementary** if no proper left or right wing of it is a two-sided word with respect to (3). A word A , two-sided with respect to the system of relations (3), will be called a **whole word with respect to (3)** if it is representable (in the free semigroup) as a product of elementary two-sided words with respect to (3). For every system of relations (3) that is closed with respect to the application of the first extension operation, the following is proved:

Lemma 1. *Every two-sided word with respect to the system of relations (3) is a whole word with respect to (3) and has a unique representation (in the free semigroup) as a product of elementary two-sided words.*

Let

$$B_1, B_2, \dots, B_s \quad (4)$$

be the set of all elementary two-sided words with respect to the system of relations (3). From Lemma 1 it follows that each A_i from the relations (3) has a unique representation $A_i \equiv \varphi_i(B_1, B_2, \dots, B_s)$, and the relations (3), including the relations (2), can be represented in the form

$$\varphi_i(B_1, B_2, \dots, B_s) = 1 \quad (i = 1, 2, \dots, k, \dots, k_1). \quad (5)$$

3. The generating operation. Let us be given a set \mathfrak{M} of relations of the system \mathfrak{B} and sets

$$\mathfrak{N}_1, \mathfrak{N}_2, \dots, \mathfrak{N}_s \quad (6)$$

of words of the system \mathfrak{B} , satisfying the following conditions:

- 1) The union of all the sets \mathfrak{N}_i contains all elementary two-sided words with respect to the system of relations \mathfrak{M} .
- 2) Any two elements of one and the same \mathfrak{N}_i are equal in the system \mathfrak{B} .
- 3) Two sets \mathfrak{N}_i and \mathfrak{N}_j for $i \neq j$ have no common elements.
- 4) The system of relations \mathfrak{M} is closed with respect to the first extension operation.
- 5) The set \mathfrak{M} consists of those and only those relations which are obtained from certain relation schemes

$$\varphi_i(x_1, x_2, \dots, x_s) = 1 \quad (i = 1, 2, \dots, k, \dots, k_1) \quad (7)$$

as the result of independently replacing all occurrences of each x_j by arbitrary elements of the set \mathfrak{N}_j ; moreover, two different occurrences of the same x_j may be replaced either by one and the same element or by different elements of the corresponding set \mathfrak{N}_j .

- 6) It is possible to choose substitutions for the variables x_1, x_2, \dots, x_s so that the schemes (7), for $i = 1, 2, \dots, k$, give the relations (2) of the system \mathfrak{B} .

Consider the group F , given by generators b_1, b_2, \dots, b_s and nontrivial defining relations

$$\varphi_i(b_1, b_2, \dots, b_s) = 1 \quad (i = 1, 2, \dots, k_1). \quad (8)$$

By the hypothesis of Theorem 1, the identity problem is decidable in F , since the lengths of the left-hand sides of the relations (8) are $\leq l$.

If E is an integral word with respect to the system of relations \mathfrak{M} , then by Lemma 1 it has a unique decomposition into elementary two-sided factors. The **image** $f(E)$ of the integral word E is the result of replacing, in this decomposition, each elementary two-sided word from the set \mathfrak{M}_j by the letter b_j . The image of any integral word is a word in the positive alphabet of the group F . If T is a word in the positive alphabet of the group F , then by a **preimage** $f^{-1}(T)$ of the word T we shall mean any integral word whose image is T .

Let A be an arbitrary word of the system \mathfrak{B} , and let

$$A \equiv XEY,$$

where E is an integral word with respect to the system of relations \mathfrak{M} , and X, Y are certain words, possibly empty. Denote by T_1 the image of the word E . Let ρ be the length of the word A . Using the algorithm that solves the identity

problem in the group F , we find all possible words in the positive alphabet of the group F

$$T_1, T_2, \dots, T_m, \quad (9)$$

which are equal in F to the word T_1 and have length $\leq \rho$. Every word

$$B \equiv Xf^{-1}(T_i)Y,$$

obtained for some choice of an integral subword E of the word A , a word T_i from the set (9) corresponding to this E , and a preimage $f^{-1}(T_i)$, will be called a word **directly generated from** A , if the length of B does not exceed ρ .

We shall say that a word B is **generated from** A if one can indicate such a finite sequence of words

$$A_1, A_2, \dots, A_j, \dots, A_r$$

that

$$A_1 \equiv A, \quad A_r \equiv B,$$

and A_i is directly generated from A_{i-1} for

$$i = 2, 3, \dots, r.$$

Since every word generated from a fixed word A of length ρ has length $\leq \rho$, the set $\pi(A)$ of all words generated from A is finite, and we have an algorithm for finding the set $\pi(A)$ from the given word A .

4. The second operation of extension of a system of defining relations

will be applied to systems of relations satisfying all six conditions indicated in item 3. Using the operation of generation, for each \mathfrak{M}_j we find the set \mathfrak{M}'_j of all those words that are generated from words of the set \mathfrak{M}_j . Then we merge those \mathfrak{M}'_j that have a nonempty intersection. In doing so, if we merge two sets \mathfrak{M}'_i and \mathfrak{M}'_j , then in all functions

$$\varphi_i(x_1, x_2, \dots, x_s)$$

we identify the variables x_i and x_j . Having performed a finite number of such mergings and the corresponding identifications, we finally obtain a collection of sets

$$\mathfrak{M}_1^*, \mathfrak{M}_2^*, \dots, \mathfrak{M}_{s_1}^*, \quad (10)$$

where $s_1 \leq s$, and the sets \mathfrak{M}_j^* are pairwise disjoint. At the same time the functions

$$\varphi_i(x_1, x_2, \dots, x_s)$$

are transformed into

$$\psi_i(x_1, x_2, \dots, x_{s_1}).$$

We define the system of relations \mathfrak{M}^* as the collection of relations obtained from the schemes of relations

$$\varphi_i^*(x_1, x_2, \dots, x_{s_1}) = 1 \quad (i = 1, 2, \dots, k_1)$$

as a result of replacing, independently of one another, all occurrences of each x_j by arbitrary elements of the corresponding set \mathfrak{M}_j^* . The system of relations \mathfrak{M}^* and the collection of sets (10) will be called the **result of applying the second operation of extension** to the system of relations \mathfrak{M} and the collection of sets (6). If $s_1 = s$ and every \mathfrak{M}_j^* coincides with \mathfrak{M}_j , then \mathfrak{M}^* will also coincide with \mathfrak{M} . In this case we shall say that the system of relations \mathfrak{M} is **closed with respect to the second operation of extension**.

5. The third operation of extension is applied to the result \mathfrak{M}^* of applying the second operation of extension, although it uses

the generation operation defined on the basis of the system of relations \mathfrak{M} and the corresponding group F . For each proper left (right) wing C of each relation of the system \mathfrak{M}^* , we find the set $\pi(C)$, if C is not a word two-sided with respect to the system \mathfrak{M} . We run through all possible pairs $[\pi(C_1), \pi(C_2)]$, where C_1 is the left wing of some relation and C_2 is the right wing of some relation. If $\pi(C_1)$ and $\pi(C_2)$ have a nonempty intersection, then we declare the elements C_1 and C_2 two-sided. To the system of relations \mathfrak{M}^* we then add all relations obtained by a cyclic permutation of C_1 (C_2) in all those relations in which C_1 (C_2) is the left (right) wing. The system of relations \mathfrak{M}^* , together with the totality of the sets (10), is called **closed with respect to the third extension operation** if, for none of the indicated pairs, the sets $\pi(C_1)$ and $\pi(C_2)$ intersect.

6. The three extension operations described are applied in the following order: first the first is applied until a system of relations closed with respect to the first extension operation is obtained; then the second is applied, and immediately after it the third extension operation; after this everything begins again from the beginning, and so on. It is established that after a finite number of steps a system of relations will be obtained that is closed with respect to the application of each of these three operations and satisfies all six conditions of item 3. For such a system of relations the fundamental lemma is proved, one of whose assertions states:

Two words X and Y are equal in the system \mathfrak{B} if and only if the corresponding sets of words $\pi(X)$ and $\pi(Y)$ have a nonempty intersection.

From this we obtain an algorithm solving the identity problem in the associative system \mathfrak{B} .

7. In ⁽²⁾ the solvability of the identity problem was established for an associative system given by a single irreducible defining relation $A = B$, where A, B are nonempty words. From Theorem 1 with $k = 1$, on the basis of B. Magnus' s result ⁽¹⁾ on the solvability of the identity problem in any group with one defining relation, it follows:

Theorem 2. *In an associative system with one defining relation of the form $A = 1$, the identity problem is solvable.*

In connection with Theorem 1 the question arises: can there exist associative systems of deficiency l with an unsolvable identity problem? The following

theorem gives an answer to this question.

Theorem 3. *For every natural $l \geq 3$ one can construct an associative system of deficiency l in which the identity problem is unsolvable. There exists an algorithm solving the identity problem in every associative system of deficiency 2.*

An associative system given by defining relations

$$A_i = B_i \quad (i = 1, 2, \dots, m)$$

is called homogeneous if the length of each A_i is equal to the length of B_i . In homogeneous associative systems the identity problem is solved very simply. From Theorem 3 it follows:

Theorem 4. *Whatever nonempty word A may be, one can construct a homogeneous associative system \mathfrak{A} such that in the system \mathfrak{A}_1 , obtained from \mathfrak{A} by adding the new relation $A = 1$, the identity problem is unsolvable.*

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
4 VII 1960

CITED LITERATURE

¹ W. Magnus, *UMN*, vol. 8, 365 (1940).

² S. I. Adian, *DAN*, 138, No. 2, 6 (1960).

Note: Figure translations are in progress. See original paper for figures.

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